

MathSolver – user guide

What MathSolver can solve, how to enter a problem and what the result looks like.

MathSolver solves problems from a photo or typed in, and for every result it shows the working the way it is done at school – step by step, with explanations and, where it makes sense, a graph.

Each chapter covers one type of problem: what to expect from the solution, how to enter it, and examples of real output from the app. Known limitations, if there are any, are listed at the end of the chapter.

Contents

1. Entering a problem
2. Linear equations and inequalities
3. Quadratic equations
4. Higher-degree equations
5. Rational equations
6. Radical equations
7. Exponential and logarithmic equations and inequalities
8. Absolute value
9. Trigonometric equations and inequalities
10. Inequalities and systems of inequalities
11. Systems of equations
12. Equations and inequalities with a parameter
13. Simplifying expressions
14. Limits
15. Derivatives
16. Function analysis
17. Taylor polynomial
18. Indefinite integrals
 - Basic integrals and substitution
 - Integration by parts
 - Rational functions

Trigonometric functions

Hyperbolic functions

Roots

19. Definite integrals and their applications

Definite integral

Improper integrals

Area under a curve and between curves

Arc length

Volume of a solid of revolution

Surface area of a solid of revolution

Parametric curves and curves in polar coordinates

Curves given by an equation

20. Infinite series

Numerical series

Power series

Taylor series

21. Fourier series

22. Differential equations

23. Recurrences and generating functions

24. Applications of differential equations

25. Coming soon

Entering a problem

You can take a photo of the problem – crop the photo to just the problem before sending it. The app recognizes it and shows a preview; if the transcription is wrong, correct it in the field below the preview. The same field lets you type a problem directly.

If the input is an expression without an equals sign, a chip lets you choose what to do with it: simplify, expand, factor, split into partial fractions, differentiate, analyze the function, compute a Taylor polynomial, integrate, sum a series or compute a Fourier series. Set the “Focus” option on the home screen to what you are practising – it preselects a suitable chip.

Equations and inequalities are solved straight away. Where there is more than one method (the method for a system, for absolute values, differentiating from the definition), you choose it with a chip; the result is recomputed at once. Equations and inequalities are solved in the real numbers; the “Solve in” chip picks $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}^+$ instead. For a limit, a chip picks whether an indeterminate form is resolved by transforming it or straight away by l'Hôpital's rule.

How to enter it

$x^2 + 3x - 4$	Powers are written with ^; multiplication by a number or a variable can be left out (3x, 2xy).
$(x+1)/(x-2)$	Fractions with a slash; put a numerator or denominator that is not a single number or letter in brackets.
$\sqrt{x+1}$, $x^{1/3}$	Square root sqrt(...), other roots as a power with a fractional exponent.
$ 2x - 1 $	Absolute value between vertical bars. If the input can be read in more than one way ($ x y z $), the app offers the readings to choose from; to make it unambiguous, put the absolute value in brackets: $2x(x - 1)$.
$\ln x$, $\log x$, $\log_2 x$	$\ln$ is the natural logarithm, $\log$ (or $\lg$) the common (base 10) one, $\log_2$ the base-2 one.
$\sin x$, $\cos 2x$, $\tan x$, $\cot x$	Trigonometric functions; an argument without brackets is the term right after the function ($\cos 2x = \cos(2x)$).

$\sinh x, \cosh x, \tanh x, \operatorname{arsinh} x$	Hyperbolic functions $\sinh, \cosh, \tanh, \coth$ and their inverses $\operatorname{arsinh}, \operatorname{arcosh}, \operatorname{artanh}, \operatorname{arcoth}$ ($\operatorname{asinh}, \operatorname{arcsinh}, \dots$ accepted too).
$\sin \alpha = 1/2, \sin(x + 30^\circ)$	An angle in degrees: a Greek letter as the unknown ($\alpha, \beta, \dots$ or α itself), or degrees on a number; Latin letters are in radians.
$\sin x = 1/2, x \in \langle 0; 2\pi \rangle$	Roots in an interval – after a comma or on the next line ($x \in \langle 0; 2\pi \rangle$) or $0 \leq x < 2\pi$.
π, e, ∞	The number π , Euler's number e , infinity ∞ .
$x \sin x, 2x(x+1)$	Put a space between a variable and a function ($x \sin x$) – $x \sin x$ is read as a product of letters.
$2x + 3y = 7$ $x - y = 1$	Write a system of equations or inequalities line by line, one per line.
$(3+4i)/(1-2i)$	Complex numbers: i is the imaginary unit (in expressions and equations).
$x^2 < 10, x \in \mathbb{Z}$	The number set after a comma: $x \in \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}^+$ (or $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}^+$, “in $\mathbb{N}$ ”); $\mathbb{N}$ without zero.

Known limitations

- The result is given in exact form (fractions, roots, π) with an approximate value; a purely numerical mode is not available yet.

Linear equations and inequalities

The app rearranges a linear equation the way it is done at school – it expands the brackets, moves the unknown to one side and divides. For an inequality it watches for the sign flip when multiplying by a negative number and writes the result as an interval; the graph shows both sides as lines and marks the solution on the x-axis.

Examples

An equation with brackets.

Input: $3(x - 2) = 2x + 5$

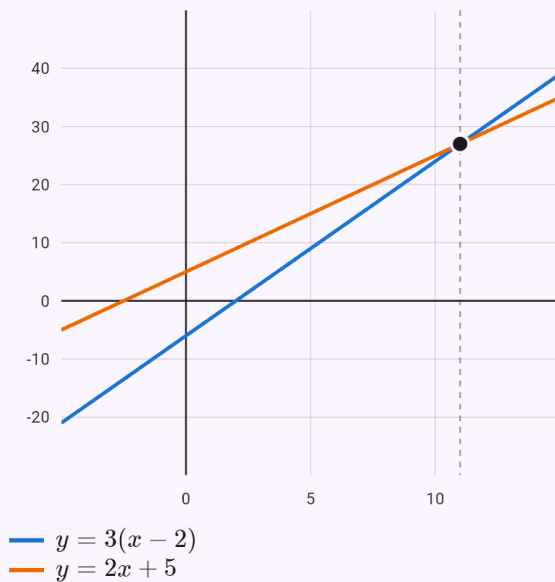
Result

$$x = 11$$

Steps

- Move everything to one side: $x - 11 = 0$
- Keep the terms with the unknown on the left, move the rest to the right: $x = 11$

Graph



Intersections of the graphs:

- (11, 27)

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

An inequality – dividing by a negative number flips the sign.

Input: $2x - 3 < 5x + 6$

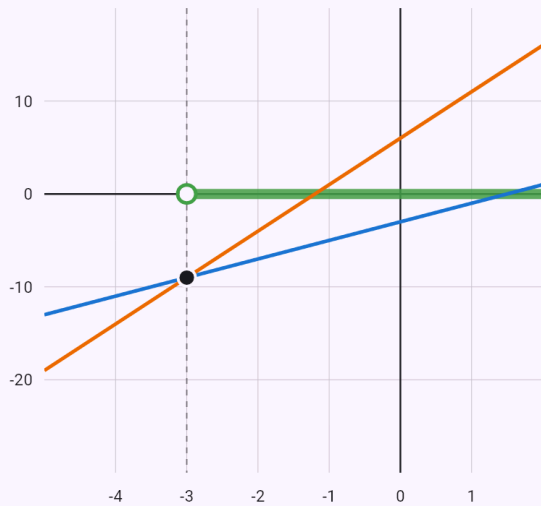
Result

$$x \in (-3, \infty)$$

Steps

- Keep the terms with the unknown on the left, move the rest to the right: $-3x < 9$
- Divide by the negative number -3 , the inequality sign flips: $x > -3$

Graph



$y = 2x - 3$
 $y = 5x + 6$

 Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

$\bullet (-3, -9)$

Quadratic equations

The app brings a quadratic equation to the form $ax^2 + bx + c = 0$ and solves it with the discriminant and the quadratic formula; it also recognizes a double root. If the equation has no real solution, it says so, and the “Show Complex Solution” button shows the complex roots.

Examples

Two real roots.

Input: $x^2 - 5x + 6 = 0$

Result

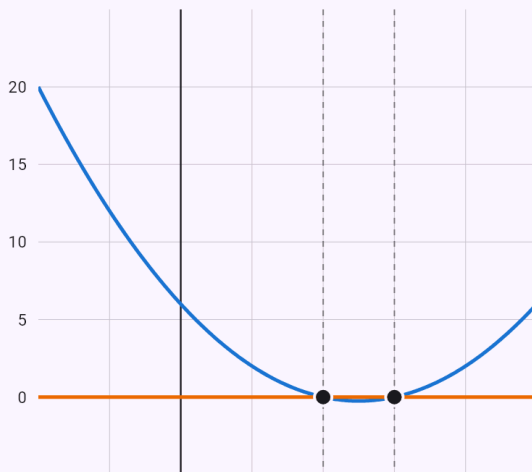
$$x_1 = 3$$

$$x_2 = 2$$

Steps

- Compute the discriminant:
 $D = b^2 - 4ac = (-5)^2 - 4 \cdot 1 \cdot 6 = 1$
- Substitute into the quadratic formula:
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{5 \pm 1}{2}$$

Graph



-1 0 1 2 3 4

— $y = x^2 - 5x + 6$
— $y = 0$

Intersections of the graphs:

• $(2, 0)$ • $(3, 0)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

A negative discriminant – complex roots.

Input: $x^2 + 2x + 5 = 0$

Show Complex Solution

Result

$$x_1 = -1 + 2i$$

$$x_2 = -1 - 2i$$

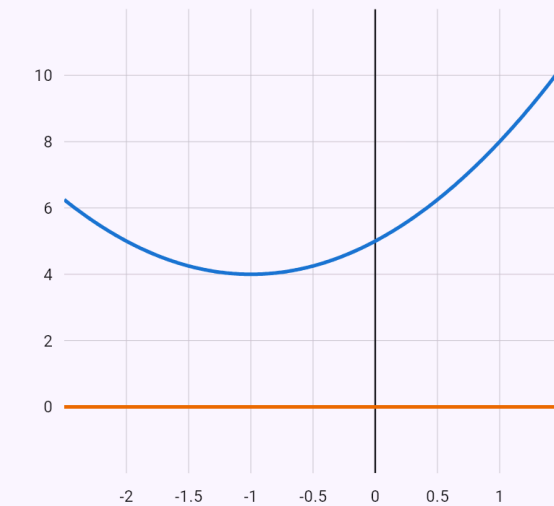
Steps

- Compute the discriminant:
 $D = b^2 - 4ac = 2^2 - 4 \cdot 1 \cdot 5 = -16$
- The discriminant is negative – over the complex numbers: $\sqrt{D} = \sqrt{-16} = 4i$

- Substitute into the quadratic formula:

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-2 \pm 4i}{2}$$

Graph



— $y = x^2 + 2x + 5$
— $y = 0$

The graphs don't intersect – the equation has no real solution.

Higher-degree equations

The app turns a biquadratic equation into a quadratic one by substitution. A general higher-degree equation is factored with synthetic division (Horner's scheme): roots are looked for among the divisors of the constant term and each root found is divided out until a quadratic equation is left.

Examples

A biquadratic equation by the substitution $y = x^2$.

Input: $x^4 - 5x^2 + 4 = 0$

Result

$$x_1 = -2$$

$$x_2 = -1$$

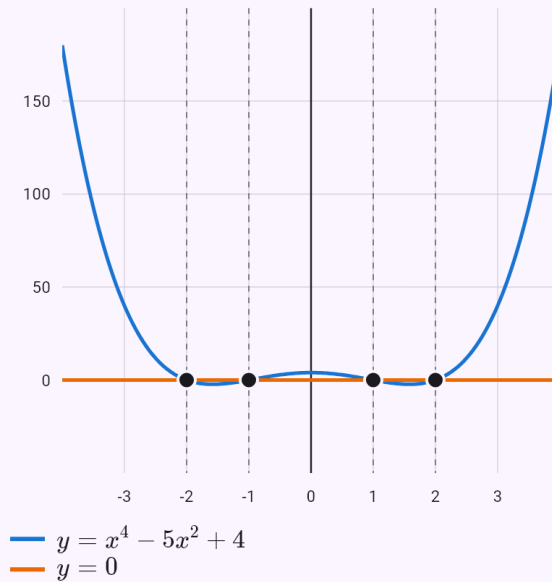
$$x_3 = 1$$

$$x_4 = 2$$

Steps

- The substitution $u = x^2$ turns the equation into a quadratic one: $u^2 - 5u + 4 = 0$
- Compute the discriminant:
 $D = b^2 - 4ac = (-5)^2 - 4 \cdot 1 \cdot 4 = 9$
- Substitute into the quadratic formula:
$$u_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{5 \pm 3}{2}$$
- Back to the original unknown: $x^2 = 1 \Rightarrow x = \pm 1$
- Back to the original unknown: $x^2 = 4 \Rightarrow x = \pm 2$

Graph



Intersections of the graphs:

• $(-2, 0)$ • $(-1, 0)$ • $(1, 0)$ • $(2, 0)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

Synthetic division – roots among the divisors of the constant term.

Input: $x^3 - 6x^2 + 11x - 6 = 0$

Result

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 3$$

Steps

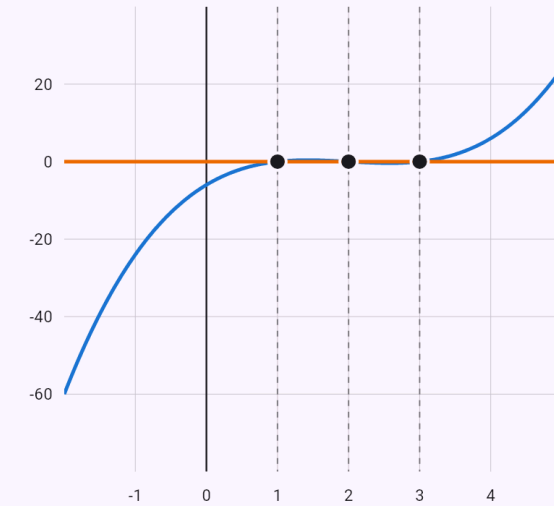
- Rational roots can only be divisors of the constant term
 $-6: \pm 1, \pm 2, \pm 3, \pm 6$
- Horner's scheme confirms the root $x = 1$ and divides the polynomial by it:

$$\begin{array}{r|rrrr} & 1 & -6 & 11 & -6 \\ 1 & 1 & -5 & 6 & 0 \end{array}$$

$$P(x) = (x - 1)(x^2 - 5x + 6)$$

- Compute the discriminant:
 $D = b^2 - 4ac = (-5)^2 - 4 \cdot 1 \cdot 6 = 1$
- Substitute into the quadratic formula:
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{5 \pm 1}{2}$$

Graph



— $y = x^3 - 6x^2 + 11x - 6$
— $y = 0$

Intersections of the graphs:

• $(1, 0)$ • $(2, 0)$ • $(3, 0)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

Rational equations

For an equation with the unknown in a denominator the app first states the conditions, then multiplies the equation by the common denominator and solves it. It compares the roots with the conditions and discards those that do not satisfy them.

Examples

Conditions, common denominator, a quadratic equation.

Input: $(x+3)/(x-1) = 2/(x-1) + x$

Result

$$x_1 = 1 - \sqrt{2}$$

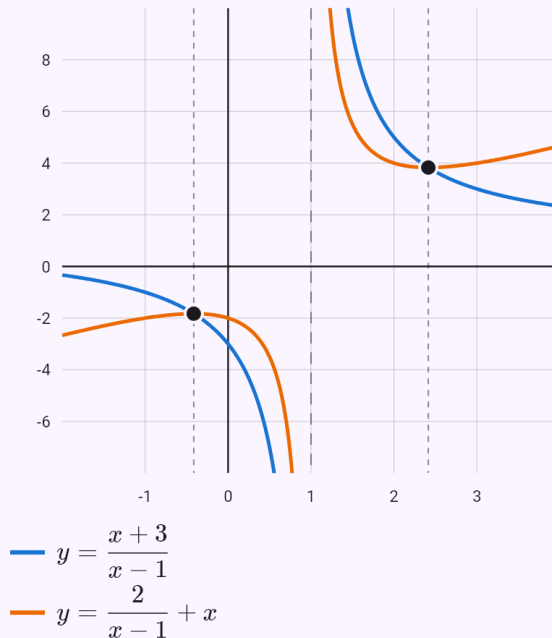
$$x_2 = 1 + \sqrt{2}$$

Conditions: $x \neq 1$

Steps

- Conditions – a denominator must not be zero: $x \neq 1$
- Multiply both sides by the denominator $x - 1$:
 $x + 3 = x^2 - x + 2$
- Move everything to one side: $x^2 - 2x - 1 = 0$
- Compute the discriminant:
 $D = b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot (-1) = 8$
- Substitute into the quadratic formula:
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{2 \pm 2\sqrt{2}}{2}$$
- The root $x = 1 - \sqrt{2}$ satisfies the conditions.
- The root $x = 1 + \sqrt{2}$ satisfies the conditions.

Graph



Intersections of the graphs:

$$\bullet \left(1 - \sqrt{2}, 1 - 2\sqrt{2}\right) \quad \bullet \left(1 + \sqrt{2}, 1 + 2\sqrt{2}\right)$$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

Radical equations

The app solves an equation with the unknown under a root by isolating the root and squaring. Squaring can add roots that do not satisfy the equation – so a check always follows and the roots that fail it are discarded.

Examples

Squaring and a check – one of the roots does not satisfy the equation.

Input: $\sqrt{x+5} = x - 1$

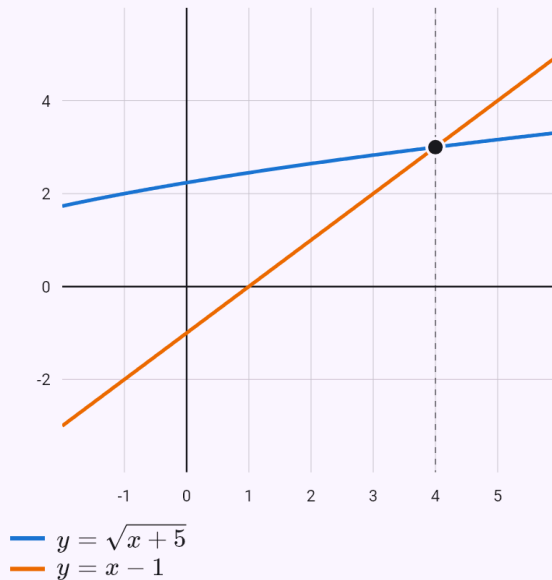
Result

$$x = 4$$

Steps

- Square both sides (not an equivalent step, so the roots are checked): $x + 5 = (x - 1)^2$
- Move everything to one side: $x^2 - 3x - 4 = 0$
- Compute the discriminant:
 $D = b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot (-4) = 25$
- Substitute into the quadratic formula:
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{3 \pm 5}{2}$$
- Check for $x = -1$: LHS = 2, RHS = -2, LHS $\neq$ RHS – the root does not work.
- Check for $x = 4$: LHS = 3, RHS = 3, LHS = RHS – the root works.

Graph



Intersections of the graphs:

- $(4, 3)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

Known limitations

- Radical inequalities are not solved yet.

Exponential and logarithmic equations and inequalities

The app solves an exponential equation by converting to the same base, by taking logarithms or by substitution; a logarithmic one by combining the logarithms by the rules and removing the logarithm, always with the conditions. Inequalities are solved through monotonicity – for a base less than 1 the sign flips.

Equations with different bases: $a^x + b^x = c^x$ with bases from two primes ($25^x + 10^x = 4^x$) are divided by a power and turned quadratic by $t = (5/2)^x$; logarithms of different bases are converted to one ($\log_5 x = \log_2 x / \log_2 5$).

Logarithms whose bases are powers of one number ($\log_2 x + \log_4(x + 2) = 2$) are converted to one base ($\log_4 a = \log_2 a / 2$), the equation multiplied and the logarithms combined. A logarithm with the unknown in its base ($\log_x 8 = 3$) is removed by the definition ($x^3 = 8$) with the conditions $x > 0$, $x \neq 1$. An equation with the unknown in both the base and the exponent ($x^{\log x} = 100$) is solved by taking logarithms and the substitution $t = \log x$. Inequalities with the substitution $t = \log(x + 1)$ and systems of logarithmic and exponential inequalities are solved like single inequalities – through monotonicity.

Equations and inequalities with hyperbolic functions are rewritten by the definition ($\sinh x = (e^x - e^{-x})/2$, ...) as exponential ones and solved by the substitution $t = e^x$; $\operatorname{arsinh} x = c$ is solved by the inverse function ($x = \sinh c$).

Examples

The substitution $t = 2^x$.

Input: $4^x - 3 \cdot 2^x - 4 = 0$

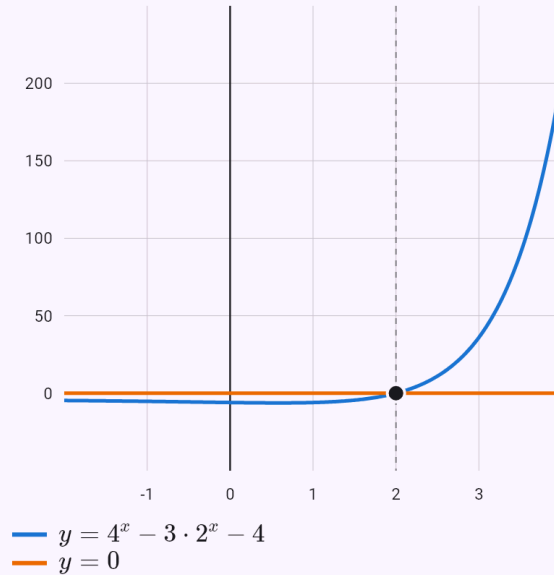
Result

$$x = 2$$

Steps

- Substitute $t = 2^x$: $t^2 - 3t - 4 = 0$
- Compute the discriminant:
 $D = b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot (-4) = 25$
- Substitute into the quadratic formula:
$$t_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{3 \pm 5}{2}$$
- The equation $2^x = -1$ has no real solution.
- Back to the original unknown: $2^x = 4 \Rightarrow x = 2$

Graph



Intersections of the graphs:

• $(2, 0)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

Combining logarithms, conditions.

Input: $\log_2(x+1) + \log_2(x-1) = 3$

Result

$$x = 3$$

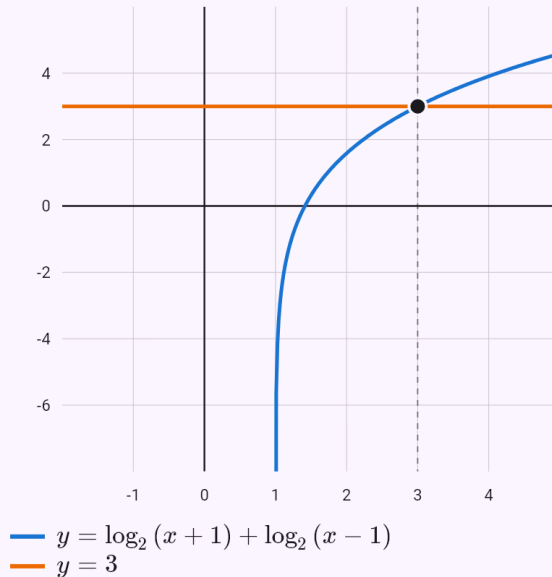
Conditions: $x > 1$

Steps

- Domain – conditions: $x > -1, x > 1 \Rightarrow x > 1$
- Combine the logarithms using the rules for logarithms:
 $\log_2((x-1)(x+1)) = 3$
- Remove the logarithm: $(x-1)(x+1) = 2^3$
- Move everything to one side: $x^2 - 9 = 0$
- Compute the discriminant:
 $D = b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot (-9) = 36$
- Substitute into the quadratic formula:

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{\pm 6}{2}$$
- The root $x = -3$ does not satisfy the condition $x > 1$, so it is not a solution.
- The root $x = 3$ satisfies the conditions.

Graph



Intersections of the graphs:

- $(3, 3)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

Different bases – divide by 4^d , substitute $t = (5/2)^d$.

Input: $25^d + 10^d = 4^d$

Result

$$d = \log_{\frac{5}{2}} \left(\frac{-1 + \sqrt{5}}{2} \right)$$

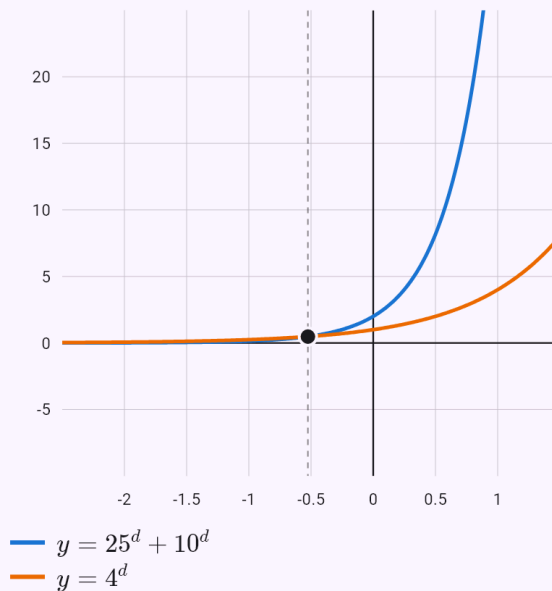
Steps

- We divide the equation by the positive 4^d :
 $\left(\frac{5}{2}\right)^{2d} + \left(\frac{5}{2}\right)^d = 1$
- Substitute $t = \left(\frac{5}{2}\right)^d$: $t^2 + t - 1 = 0$
- Compute the discriminant:
 $D = b^2 - 4ac = 1^2 - 4 \cdot 1 \cdot (-1) = 5$
- Substitute into the quadratic formula:

$$t_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{5}}{2}$$
- The equation $\left(\frac{5}{2}\right)^d = \frac{-1 - \sqrt{5}}{2}$ has no real solution.
- Back to the original unknown

$$\left(\frac{5}{2}\right)^d = \frac{-1 + \sqrt{5}}{2} \Rightarrow d = \log_{\frac{5}{2}} \left(\frac{-1 + \sqrt{5}}{2} \right)$$

Graph



Intersections of the graphs:

- $\left(\log_{\frac{5}{2}} \left(\frac{-1 + \sqrt{5}}{2} \right), 0.4829 \right)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of d .

A base less than 1 – the inequality sign flips.

Input: $(1/2)^x > 8$

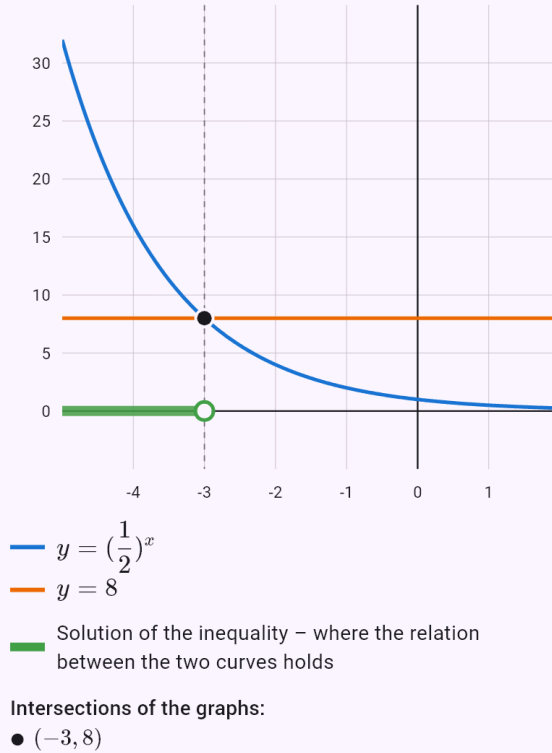
Result

$$x \in (-\infty, -3)$$

Steps

- Write both sides as powers with the same base:
 $2^{-x} > 2^3$
- The exponential function with base 2 is increasing – the exponents satisfy the inequality with the same sign:
 $-x > 3$
- Keep the terms with the unknown on the left, move the rest to the right: $-x > 3$
- Divide by the negative number -1 , the inequality sign flips: $x < -3$
- The solution of this inequality is $x \in (-\infty, -3)$

Graph



Hyperbolic tangent by the definition – substitution $t = e^{2x}$.

Input: $\tanh x = 1/2$

Result

$$x = \frac{\ln 3}{2}$$

Steps

- Substituting the definitions

$$\tanh \alpha = \frac{e^\alpha - e^{-\alpha}}{e^\alpha + e^{-\alpha}} = \frac{e^{2\alpha} - 1}{e^{2\alpha} + 1}$$

$$\frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1}{2} \Rightarrow \frac{e^{2x} - 3}{2e^{2x} + 2} = 0 \Rightarrow e^{2x} - 3 = 0$$

- Substitute $t = e^{2x}$: $t - 3 = 0$
- Solve: $t = 3$
- Back to the original unknown: $e^{2x} = 3 \Rightarrow x = \frac{\ln 3}{2}$

The unknown in the base – by the definition; the root -1 fails the conditions.

Input: $\log_x (2x+3) = 2$

Result

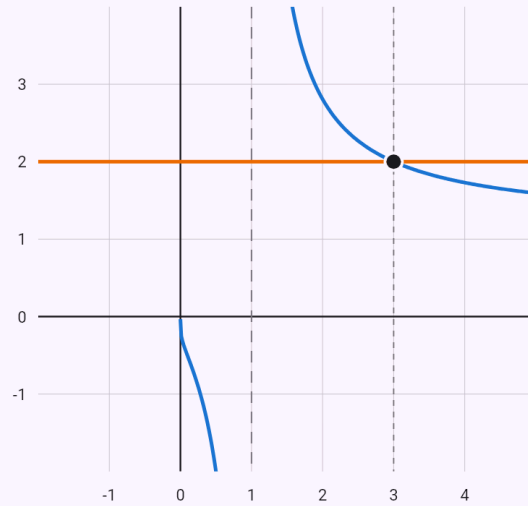
$$x = 3$$

Conditions: $x > 0$, $x \neq 1$

Steps

- Domain – conditions: $x > 0$, $x \neq 1$
- Remove the logarithm: $x^2 = 2x + 3$
- Move everything to one side: $x^2 - 2x - 3 = 0$
- Compute the discriminant:
 $D = b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot (-3) = 16$
- Substitute into the quadratic formula:
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{2 \pm 4}{2}$$
- The root $x = -1$ does not satisfy the condition $x > 0$, $x \neq 1$, so it is not a solution.
- The root $x = 3$ satisfies the conditions.

Graph



— $y = \log_x (2x + 3)$
— $y = 2$

Intersections of the graphs:

• $(3, 2)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

The unknown in the base and the exponent – take logarithms, substitute $t = \log x$.

Input: $x^{(\log x)} = 100$

Result

$$x_1 = 10^{-\sqrt{2}}$$

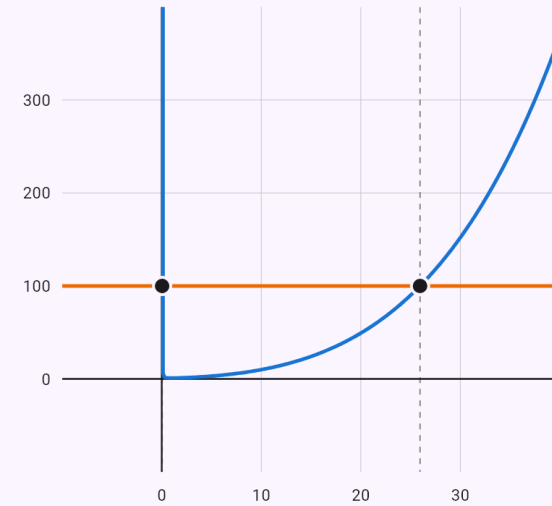
$$x_2 = 10^{\sqrt{2}}$$

Conditions: $x > 0$

Steps

- Take the logarithm of both sides:
 $\log(x^{\log x}) = 2 \Rightarrow \log x \cdot \log x = 2$
- Domain – conditions: $x > 0$
- Substitute $t = \log x$: $t^2 - 2 = 0$
- Compute the discriminant:
 $D = b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot (-2) = 8$
- Substitute into the quadratic formula:
 $t_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{\pm 2\sqrt{2}}{2}$
- Back to the original unknown:
 $\log x = -\sqrt{2} \Rightarrow x = 10^{-\sqrt{2}}$
- Back to the original unknown:
 $\log x = \sqrt{2} \Rightarrow x = 10^{\sqrt{2}}$
- The root $x = 10^{-\sqrt{2}}$ satisfies the conditions.
- The root $x = 10^{\sqrt{2}}$ satisfies the conditions.

Graph



$y = x^{\log x}$
 $y = 100$

Intersections of the graphs:

• $(10^{-\sqrt{2}}, 100)$ • $(10^{\sqrt{2}}, 100)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

Known limitations

- Logarithms of different bases that aren't powers of one number ($\log_2 x + \log_3(x + 1) = 2$) are solved only numerically.
- Hyperbolic equations leading to a 4th-degree equation in e^x ($\cosh^2 x - 3 \sinh x = 3$) are solved only numerically; inequalities with inverse hyperbolic functions are not supported yet.

Absolute value

The app solves equations and inequalities with absolute values in two ways, which you choose with a chip: “By the definition” – a shortcut that writes out $|a| = b$ or $|a| < b$ – and “Zero points” – splitting the number line into intervals on which each absolute value has a fixed sign. Without a choice it picks the more suitable one itself.

Nested absolute values ($||x| - 2| = 1$) are solved the same way: by the definition step by step from the outside in, by zero points from the inside out – first the zero points of the inner absolute values, then the remaining absolute value on each interval.

An inequality $|f| < |g|$ (just an absolute value on each side) can also be solved with the “By squaring” chip: both sides are non-negative, so it is squared and solved as a quadratic one. An expression with absolute values alone ($|x - 1| + |x + 2|$) is simplified by pieces – zero points, a sign chart and the expression without absolute values on each interval.

Examples

One absolute value – written out by the definition.

Input: $|2x - 1| = 5$

Result

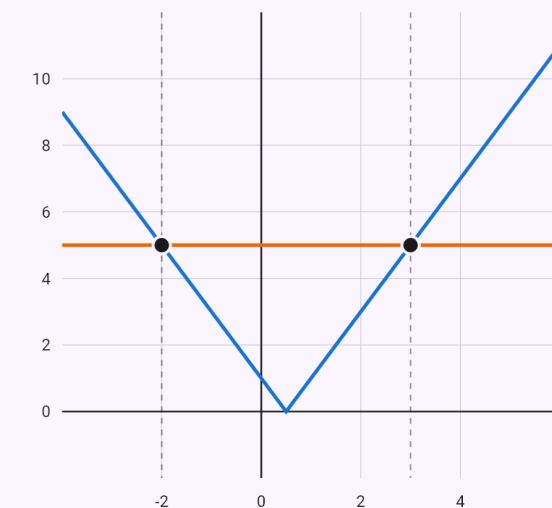
$$x_1 = -2$$

$$x_2 = 3$$

Steps

- There is only one absolute value and a number on the other side – it is enough to use the definition.
- By the definition, $|2x - 1| = 5$ holds exactly when $2x - 1 = 5$ or $2x - 1 = -5$
- Solve: $2x - 1 = 5 \Rightarrow x = 3$
- Solve: $2x - 1 = -5 \Rightarrow x = -2$

Graph



$y = |2x - 1|$
 $y = 5$

Intersections of the graphs:

• $(-2, 5)$ • $(3, 5)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

Two absolute values – the zero-points method.

Input: $|x - 1| + |x + 2| < 5$

Result

$$x \in (-3, 2)$$

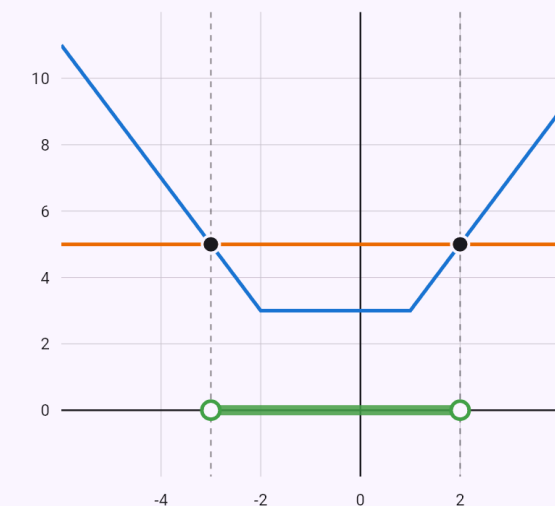
Steps

- Remove the absolute values by the zero point method: the zero points split the number line into intervals, on each of them the expressions inside keep their sign.
- Zero points of the expressions inside the absolute values: $x - 1 = 0 \Rightarrow x = 1$, $x + 2 = 0 \Rightarrow x = -2$
- Signs on the individual intervals:

	$(-\infty, -2)$	$[-2, 1)$	$[1, \infty)$
$x - 1$	-	-	+
$x + 2$	-	+	+
- On the interval $(-\infty, -2)$, remove the absolute values by the signs: $(1 - x) + (-x - 2) < 5$
- Solve: $-2x - 1 < 5$
- Keep the terms with the unknown on the left, move the rest to the right: $-2x < 6$
- Divide by the negative number -2 , the inequality sign flips: $x > -3$
- The solution of this inequality is $x \in (-3, \infty)$
- Intersection with the interval $(-\infty, -2)$: $x \in (-3, -2)$
- On the interval $[-2, 1)$, remove the absolute values by the signs: $(1 - x) + (x + 2) < 5$

- The unknown cancelled and the true statement $3 < 5$ is left – the whole interval $[-2, 1)$ is a solution.
- On the interval $[1, \infty)$, remove the absolute values by the signs: $(x - 1) + (x + 2) < 5$
- Solve: $2x + 1 < 5$
- Keep the terms with the unknown on the left, move the rest to the right: $2x < 4$
- Divide by the positive number 2, the inequality sign stays: $x < 2$
- The solution of this inequality is $x \in (-\infty, 2)$
- Intersection with the interval $[1, \infty)$: $x \in [1, 2)$
- The solution is the union of the results: $x \in (-3, 2)$

Graph



$y = |x - 1| + |x + 2|$
 $y = 5$

Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

$\bullet (-3, 5)$ $\bullet (2, 5)$

A nested absolute value – by the definition from the outside in.

Input: $||x| - 2| = 1$

Result

$$x_1 = -3$$

$$x_2 = -1$$

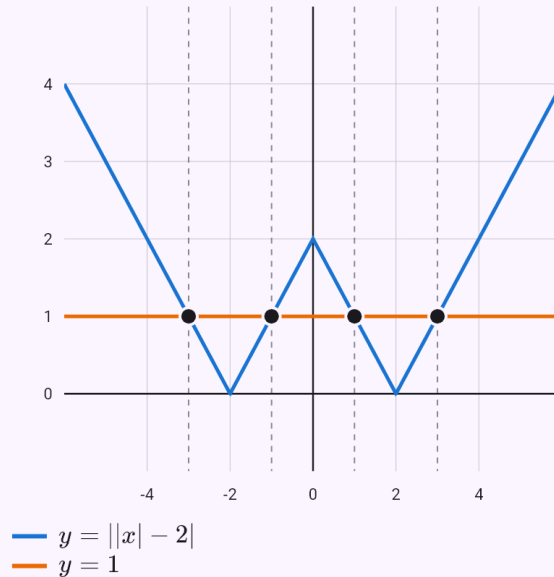
$$x_3 = 1$$

$$x_4 = 3$$

Steps

- There is only one absolute value and a number on the other side – it is enough to use the definition.
- By the definition, $||x| - 2| = 1$ holds exactly when $|x| - 2 = 1$ or $|x| - 2 = -1$
- Isolate the absolute value: $|x| = 3$
- By the definition, $|x| = 3$ holds exactly when $x = 3$ or $x = -3$
- Isolate the absolute value: $|x| = 1$
- By the definition, $|x| = 1$ holds exactly when $x = 1$ or $x = -1$

Graph



Intersections of the graphs:

$$\bullet (-3, 1) \quad \bullet (-1, 1) \quad \bullet (1, 1) \quad \bullet (3, 1)$$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

Absolute values on both sides – squaring.

Input: $|x - 1| < |2x + 3|$ By squaring

Result

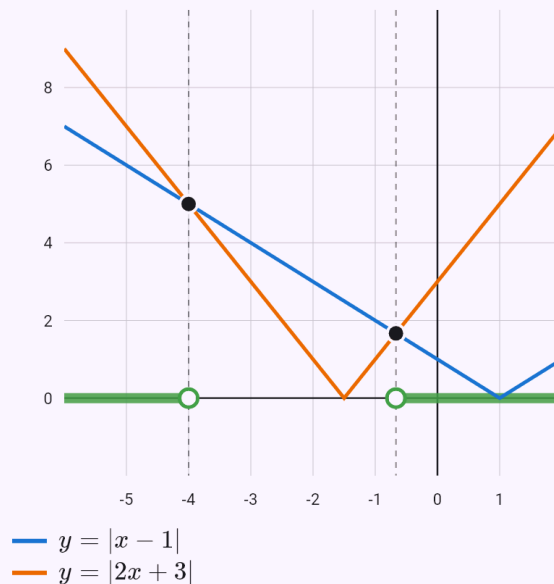
$$x \in (-\infty, -4) \cup \left(-\frac{2}{3}, \infty\right)$$

Steps

- Both sides are just absolute values – both are non-negative, so we can square the inequality (an equivalent step).
- Square both sides: $(x - 1)^2 < (2x + 3)^2$
- Move everything to one side: $-3x^2 - 14x - 8 < 0$
- Factor: $-(x + 4)(3x + 2) < 0$
- The critical points split the number line into intervals:
 $x_1 = -4$, $x_2 = -\frac{2}{3}$
- Signs on the individual intervals:

	$(-\infty, -4)$	$(-4, -\frac{2}{3})$	$(-\frac{2}{3}, \infty)$
-1	–	–	–
$x + 4$	–	+	+
$3x + 2$	–	–	+
$-(x + 4)(3x + 2)$	–	+	–

Graph



Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

$$\bullet (-4, 5) \quad \bullet \left(-\frac{2}{3}, \frac{5}{3}\right)$$

An expression with absolute values – by pieces on the intervals.

Input: $|x - 1| + |x + 2|$

Result

$$\begin{cases} -2x - 1, & x \in (-\infty, -2) \\ 3, & x \in [-2, 1) \\ 2x + 1, & x \in [1, \infty) \end{cases}$$

Steps

- Remove the absolute values by the zero point method: the zero points split the number line into intervals, on each of them the expressions inside keep their sign.
- Zero points of the expressions inside the absolute values: $x - 1 = 0 \Rightarrow x = 1$, $x + 2 = 0 \Rightarrow x = -2$

- Signs on the individual intervals:

	$(-\infty, -2)$	$[-2, 1)$	$[1, \infty)$
$x - 1$	-	-	+
$x + 2$	-	+	+

- On the interval $(-\infty, -2)$, remove the absolute values by the signs: $(1 - x) + (-x - 2) = -2x - 1$
- On the interval $[-2, 1)$, remove the absolute values by the signs: $(1 - x) + (x + 2) = 3$
- On the interval $[1, \infty)$, remove the absolute values by the signs: $(x - 1) + (x + 2) = 2x + 1$

Known limitations

- An expression with absolute values is simplified only on the whole number line, not on a given interval; squaring is offered only for an inequality with a single absolute value on each side.

Trigonometric equations and inequalities

The app brings an equation to basic equations such as $\sin x = a$ and solves them on the unit circle: it finds the angles of one turn and adds multiples of the period ($x = \pi/6 + 2k\pi$, $k \in \mathbb{Z}$). The circle is drawn right in the steps.

It prefers the known formulas – double and half angle, $\sin^2 x + \cos^2 x = 1$, sum to product –, then factoring, the substitution $t = \sin x$, dividing a homogeneous equation by $\cos^n x$ and the auxiliary angle for $a \cdot \sin x + b \cdot \cos x = c$. The result is also checked numerically over a whole period.

An unknown written as a Latin letter (x , t) is in radians, a Greek one (α , β , ...) in degrees – as is an input with the $^\circ$ sign ($\sin(x + 30^\circ)$). To get the roots in a given interval, add the interval after a comma or on the next line.

An inequality is solved by an arc of the unit circle, a more complex one by the zeros on one period; the result is intervals with the period.

Examples

Two branches of solutions, the angles on the unit circle.

Input: $\sin x = 1/2$

Result

$$x_1 = \frac{\pi}{6} + 2k\pi$$

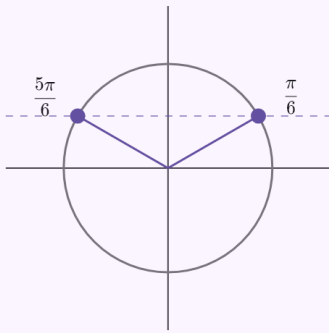
$$x_2 = \frac{5\pi}{6} + 2k\pi$$

$$k \in \mathbb{Z}$$

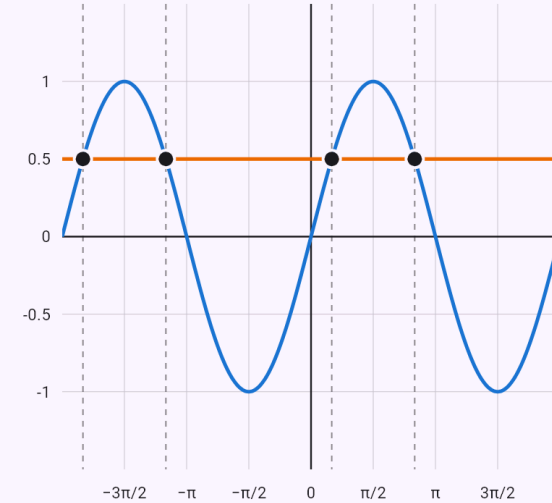
Steps

- Basic equation $\sin x = \frac{1}{2}$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{6}$, $\frac{5\pi}{6}$; adding multiples of the period gives all solutions:

$$x = \frac{\pi}{6} + 2k\pi, \quad x = \frac{5\pi}{6} + 2k\pi$$



Graph



$$\begin{aligned} \text{--- } y &= \sin x \\ \text{--- } y &= \frac{1}{2} \end{aligned}$$

Intersections of the graphs:

$$\begin{aligned} &\bullet \left(-\frac{11\pi}{6}, \frac{1}{2} \right) \quad \bullet \left(-\frac{7\pi}{6}, \frac{1}{2} \right) \quad \bullet \left(\frac{\pi}{6}, \frac{1}{2} \right) \\ &\bullet \left(\frac{5\pi}{6}, \frac{1}{2} \right) \end{aligned}$$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

The substitution $t = \sin x$ and a quadratic equation.

Input: $2\sin^2 x - \sin x - 1 = 0$

Result

$$x_1 = \frac{\pi}{2} + 2k\pi$$

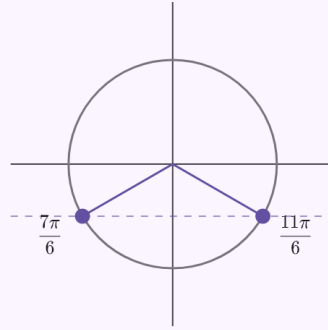
$$x_2 = \frac{7\pi}{6} + 2k\pi$$

$$x_3 = \frac{11\pi}{6} + 2k\pi$$

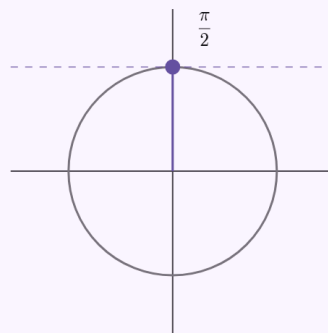
$$k \in \mathbb{Z}$$

Steps

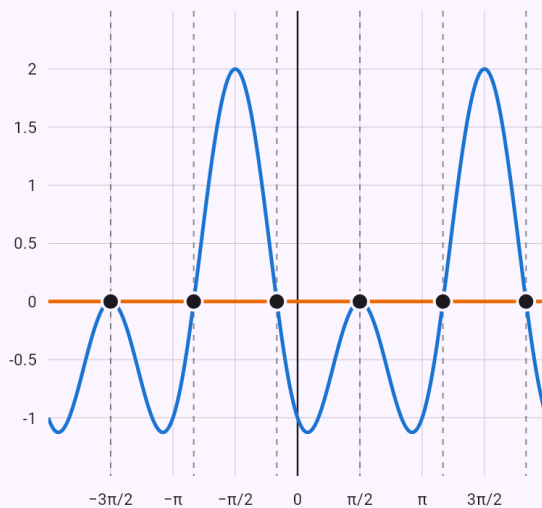
- Substitute $t = \sin x$: $2t^2 - t - 1 = 0$
- Compute the discriminant:
 $D = b^2 - 4ac = (-1)^2 - 4 \cdot 2 \cdot (-1) = 9$
- Substitute into the quadratic formula:
 $t_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 3}{4}$
- Basic equation $\sin x = -\frac{1}{2}$: on the unit circle, the angles of one turn satisfying it are $\frac{7\pi}{6}, \frac{11\pi}{6}$; adding multiples of the period gives all solutions:
 $x = \frac{7\pi}{6} + 2k\pi, x = \frac{11\pi}{6} + 2k\pi$



- Basic equation $\sin x = 1$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{2}$; adding multiples of the period gives all solutions: $x = \frac{\pi}{2} + 2k\pi$



Graph



— $y = 2\sin^2 x - \sin x - 1$
 — $y = 0$

Intersections of the graphs:

$$\bullet \left(-\frac{3\pi}{2}, 0\right) \bullet \left(-\frac{5\pi}{6}, 0\right) \bullet \left(-\frac{\pi}{6}, 0\right) \bullet \left(\frac{\pi}{2}, 0\right)$$

$$\bullet \left(\frac{7\pi}{6}, 0\right) \bullet \left(\frac{11\pi}{6}, 0\right)$$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

The formula for $\sin 2x$ and factoring.

Input: $\sin 2x = \cos x$

Result

$$x_1 = \frac{\pi}{6} + 2k\pi$$

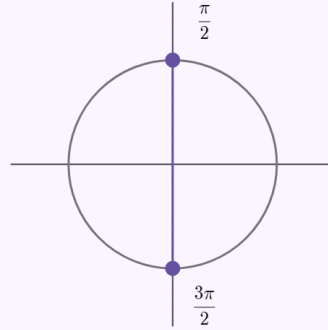
$$x_2 = \frac{\pi}{2} + k\pi$$

$$x_3 = \frac{5\pi}{6} + 2k\pi$$

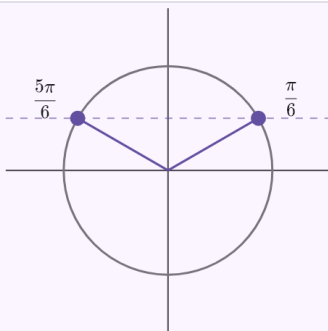
$$k \in \mathbb{Z}$$

Steps

- Move everything to one side: $\sin(2x) - \cos x = 0$
- Use the formula $\sin 2x = 2 \sin x \cos x$:
 $2 \cos x \cdot \sin x - \cos x = 0$
- Factor – a product is zero exactly when one of its factors is: $(2 \sin x - 1) \cos x = 0$
- Solve the equation $\cos x = 0$
- Basic equation $\cos x = 0$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{2}, \frac{3\pi}{2}$; adding multiples of the period gives all solutions: $x = \frac{\pi}{2} + k\pi$



- Solve the equation $2 \sin x - 1 = 0$
- Isolate the trigonometric function: $\sin x = \frac{1}{2}$
- Basic equation $\sin x = \frac{1}{2}$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{6}, \frac{5\pi}{6}$; adding multiples of the period gives all solutions:
 $x = \frac{\pi}{6} + 2k\pi, x = \frac{5\pi}{6} + 2k\pi$



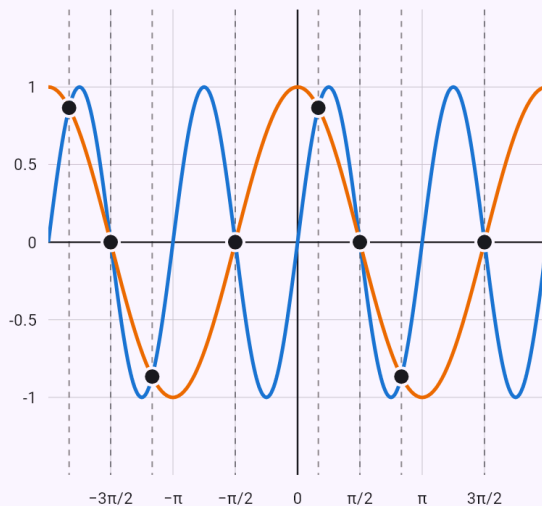
$y = \sin 2x$
 $y = \cos x$

Intersections of the graphs:

$$\begin{aligned}
 &\bullet \left(-\frac{11\pi}{6}, \frac{\sqrt{3}}{2}\right) \bullet \left(-\frac{3\pi}{2}, 0\right) \bullet \left(-\frac{7\pi}{6}, -\frac{\sqrt{3}}{2}\right) \\
 &\bullet \left(-\frac{\pi}{2}, 0\right) \bullet \left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right) \bullet \left(\frac{\pi}{2}, 0\right) \bullet \left(\frac{5\pi}{6}, -\frac{\sqrt{3}}{2}\right) \\
 &\bullet \left(\frac{3\pi}{2}, 0\right)
 \end{aligned}$$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

Graph



The auxiliary angle: $\sin x + \cos x = \sqrt{2} \cdot \sin(x + \pi/4)$.

Input: $\sin x + \cos x = 1$

Result

$$x_1 = 2k\pi$$

$$x_2 = \frac{\pi}{2} + 2k\pi$$

$$k \in \mathbb{Z}$$

Steps

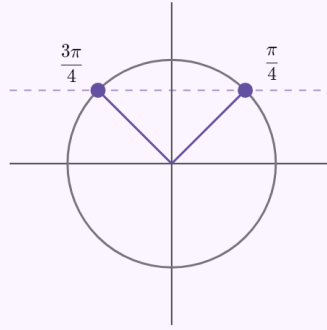
- Move everything to one side: $\cos x + \sin x - 1 = 0$

- Introduce an auxiliary angle φ :
 $\cos x + \sin x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$, where

$$\cos \varphi = \frac{\sqrt{2}}{2}, \sin \varphi = \frac{\sqrt{2}}{2} \Rightarrow \varphi = \frac{\pi}{4}$$

- Isolate the trigonometric function: $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

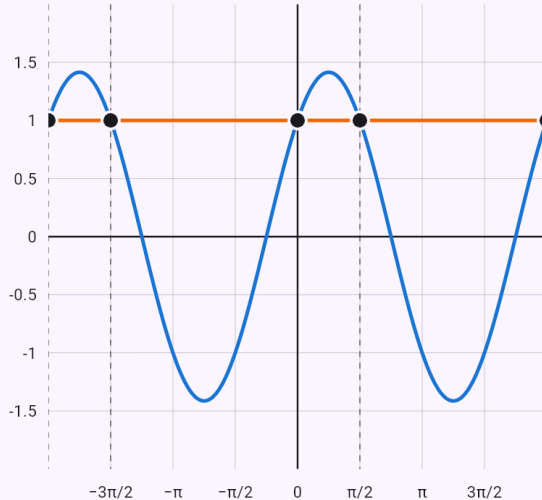
- Basic equation $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{4}$, $\frac{3\pi}{4}$; adding multiples of the period gives all solutions:
 $x + \frac{\pi}{4} = \frac{\pi}{4} + 2k\pi, x + \frac{\pi}{4} = \frac{3\pi}{4} + 2k\pi$



- Solve for the unknown

$$x + \frac{\pi}{4} = \frac{\pi}{4} + 2k\pi \Rightarrow x = 2k\pi \vee x + \frac{\pi}{4} = \frac{3\pi}{4} + 2k\pi$$

Graph



— $y = \sin x + \cos x$
 — $y = 1$

Intersections of the graphs:

- $(-2\pi, 1)$ • $\left(-\frac{3\pi}{2}, 1\right)$ • $(0, 1)$ • $\left(\frac{\pi}{2}, 1\right)$
- $(2\pi, 1)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

A Greek letter – the result in degrees.

Input: `sin alpha + cos alpha = sqrt(2)`

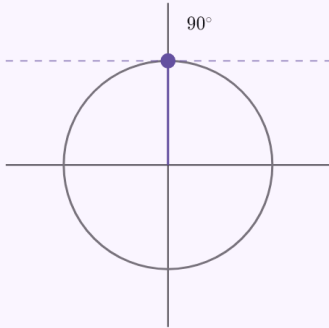
Result

$$\alpha = 45^\circ + k \cdot 360^\circ$$

$$k \in \mathbb{Z}$$

Steps

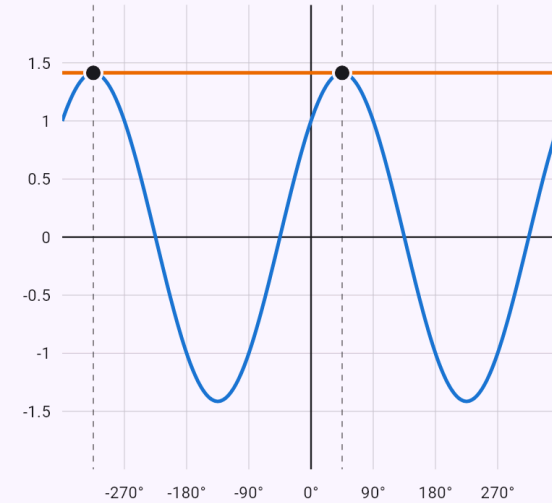
- Move everything to one side: $\cos \alpha + \sin \alpha - \sqrt{2} = 0$
- Introduce an auxiliary angle φ :
 $\cos \alpha + \sin \alpha = \sqrt{2} \sin(\alpha + 45^\circ)$, where
 $\cos \varphi = \frac{\sqrt{2}}{2}$, $\sin \varphi = \frac{\sqrt{2}}{2} \Rightarrow \varphi = 45^\circ$
- Isolate the trigonometric function: $\sin(\alpha + 45^\circ) = 1$
- Basic equation $\sin(\alpha + 45^\circ) = 1$: on the unit circle, the angles of one turn satisfying it are 90° ; adding multiples of the period gives all solutions:
 $\alpha + 45^\circ = 90^\circ + k \cdot 360^\circ$



- Solve for the unknown

$$\alpha + 45^\circ = 90^\circ + k \cdot 360^\circ \Rightarrow \alpha = 45^\circ + k \cdot 360^\circ$$

Graph



$$y = \sin \alpha + \cos \alpha$$

$$y = \sqrt{2}$$

Intersections of the graphs:

$$\bullet \left(-315^\circ, \sqrt{2}\right) \quad \bullet \left(45^\circ, \sqrt{2}\right)$$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

Only the roots in the given interval.

Input: `sin x = 1/2, x ∈ <0; 4pi)`

Result

$$x_1 = \frac{\pi}{6}$$

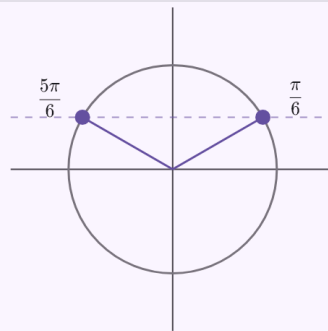
$$x_2 = \frac{5\pi}{6}$$

$$x_3 = \frac{13\pi}{6}$$

$$x_4 = \frac{17\pi}{6}$$

Steps

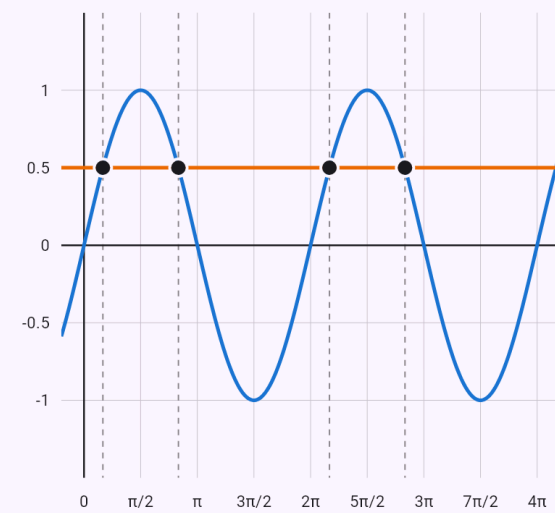
- Basic equation $\sin x = \frac{1}{2}$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{6}$, $\frac{5\pi}{6}$; adding multiples of the period gives all solutions:
 $x = \frac{\pi}{6} + 2k\pi$, $x = \frac{5\pi}{6} + 2k\pi$



- The solutions in the interval $x \in [0, 4\pi)$

$$x_1 = \frac{\pi}{6}, x_2 = \frac{5\pi}{6}, x_3 = \frac{13\pi}{6}, x_4 = \frac{17\pi}{6}$$

Graph



— $y = \sin x$

— $y = \frac{1}{2}$

Intersections of the graphs:

• $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ • $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$ • $\left(\frac{13\pi}{6}, \frac{1}{2}\right)$ • $\left(\frac{17\pi}{6}, \frac{1}{2}\right)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x .

An arc of the unit circle, an interval around zero.

Input: $\cos x > \frac{1}{2}$

Result

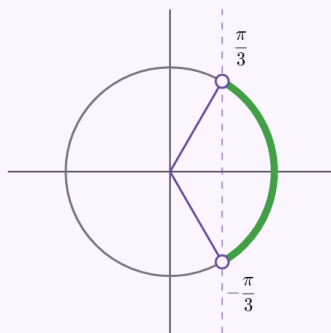
$$x \in \left(-\frac{\pi}{3} + 2k\pi, \frac{\pi}{3} + 2k\pi\right)$$

$$k \in \mathbb{Z}$$

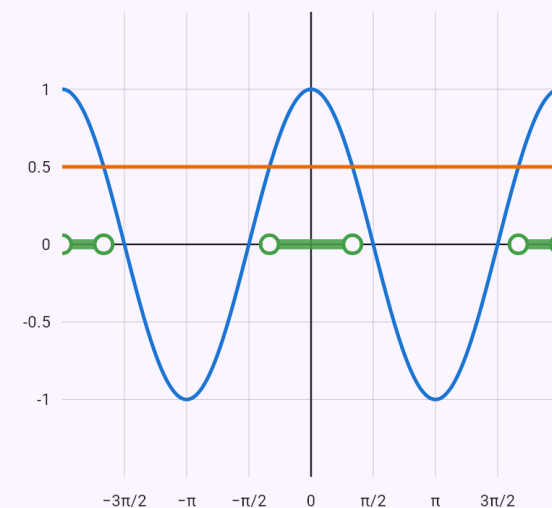
Steps

- Basic inequality $\cos x > \frac{1}{2}$: on the unit circle it holds on the highlighted arc; with the period:

$$x \in \left(-\frac{\pi}{3} + 2k\pi, \frac{\pi}{3} + 2k\pi\right)$$



Graph



— $y = \cos x$

— $y = \frac{1}{2}$

— Solution of the inequality – where the relation between the two curves holds

The zeros on a period and the sign between them.

Input: $2\sin^2 x - \sin x - 1 > 0$

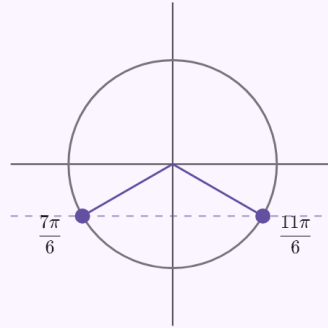
Result

$$x \in \left(\frac{7\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi \right)$$

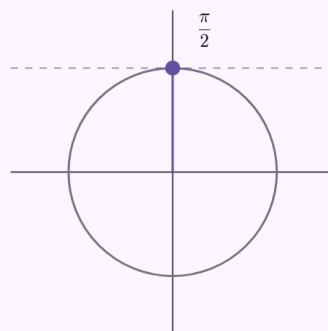
$$k \in \mathbb{Z}$$

Steps

- First solve the equation $-\sin x + 2\sin^2 x - 1 = 0$
- Substitute $t = \sin x$: $2t^2 - t - 1 = 0$
- Compute the discriminant:
 $D = b^2 - 4ac = (-1)^2 - 4 \cdot 2 \cdot (-1) = 9$
- Substitute into the quadratic formula:
 $t_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 3}{4}$
- Basic equation $\sin x = -\frac{1}{2}$: on the unit circle, the angles of one turn satisfying it are $\frac{7\pi}{6}, \frac{11\pi}{6}$; adding multiples of the period gives all solutions:
 $x = \frac{7\pi}{6} + 2k\pi, x = \frac{11\pi}{6} + 2k\pi$

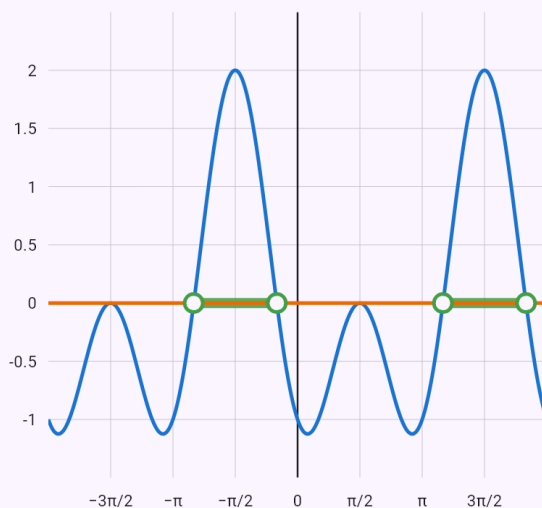


- Basic equation $\sin x = 1$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{2}$; adding multiples of the period gives all solutions: $x = \frac{\pi}{2} + 2k\pi$



- The function has the period 2π . Its zeros (and the points where it isn't defined) $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ split one period into intervals; by the sign on them, the inequality holds for:
 $x \in \left(\frac{7\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi \right)$

Graph



$y = 2\sin^2 x - \sin x - 1$
 $y = 0$

 Solution of the inequality – where the relation between the two curves holds

Known limitations

- Equations with the unknown also outside a trigonometric function ($x = \cos x$) are solved only numerically, without the period.
- A double trigonometric inequality ($-1/2 < \sin x < 1/2$) and trigonometric functions inside an absolute value are not supported yet.

Inequalities and systems of inequalities

The app solves quadratic and rational inequalities with the zero points and a sign chart. A double inequality and a system of inequalities are solved as the intersection of the solutions of the individual inequalities, shown on a number line. A system of inequalities in two unknowns is solved as a region in the plane, shaded in the graph.

In a system of inequalities in two unknowns the app also lists the region's corners – the intersections of the boundaries with exact coordinates. A system of an equation and inequalities in one unknown is solved by solving the equation and substituting its roots into the inequalities; in two unknowns the solution is the part of the equation's curve lying in the inequalities' region – highlighted in the graph, its end points computed exactly. A system can also be written on one line, separated by semicolons ($2^x > 4$; $3^x < 81$).

Examples

A quadratic inequality – zero points and signs.

Input: $x^2 - x - 6 > 0$

Result

$$x \in (-\infty, -2) \cup (3, \infty)$$

Steps

- Factor: $(x - 3)(x + 2) > 0$
- The critical points split the number line into intervals:
 $x_1 = -2$, $x_2 = 3$
- Signs on the individual intervals:

	$(-\infty, -2)$	$(-2, 3)$	$(3, \infty)$
$x - 3$	-	-	+
$x + 2$	-	+	+
$(x - 3)(x + 2)$	+	-	+

Graph



$$y = x^2 - x - 6$$

$$y = 0$$

Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

$$\bullet (-2, 0) \quad \bullet (3, 0)$$

A rational inequality – the denominator must not be zero.

Input: $(x+1)/(x-3) \leq 0$

Result

$$x \in [-1, 3)$$

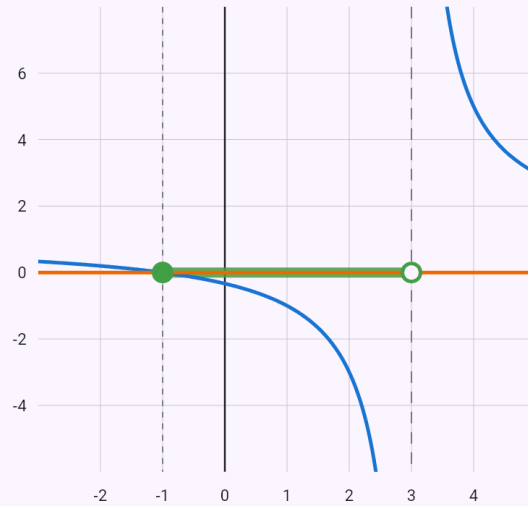
Conditions: $x \neq 3$

Steps

- Conditions – a denominator must not be zero: $x \neq 3$
- The critical points split the number line into intervals:
 $x_1 = -1, x_2 = 3$
- Signs on the individual intervals:

	$(-\infty, -1)$	$(-1, 3)$	$(3, \infty)$
$x - 3$	-	-	+
$x + 1$	-	+	+
$\frac{x+1}{x-3}$	+	-	+

Graph



$$y = \frac{x+1}{x-3}$$
$$y = 0$$

 Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

• $(-1, 0)$

A double inequality.

Input: $-1 \leq 2x + 1 < 5$

Result

$$x \in [-1, 2)$$

Steps

- Inequality 1: $-1 \leq 2x + 1$
- Keep the terms with the unknown on the left, move the rest to the right: $-2x \leq 2$
- Divide by the negative number -2 , the inequality sign flips: $x \geq -1$
- The solution of this inequality is $x \in [-1, \infty)$
- Inequality 2: $2x + 1 < 5$
- Keep the terms with the unknown on the left, move the rest to the right: $2x < 4$
- Divide by the positive number 2 , the inequality sign stays: $x < 2$
- The solution of this inequality is $x \in (-\infty, 2)$
- The system's solution is the intersection of the inequalities' solutions: $x \in [-1, 2)$

Graph



Solution of the system (intersection)



Each inequality has its own row on the number line, their intersection is in green. A filled dot belongs to the solution, a hollow one doesn't.

A system of inequalities in two unknowns – a region in the plane and its corners.

Input: $x + y \leq 4$

$x - y \geq 0$

$$y \geq 0$$

Result

$$y \leq 4 - x$$

$$y \leq x$$

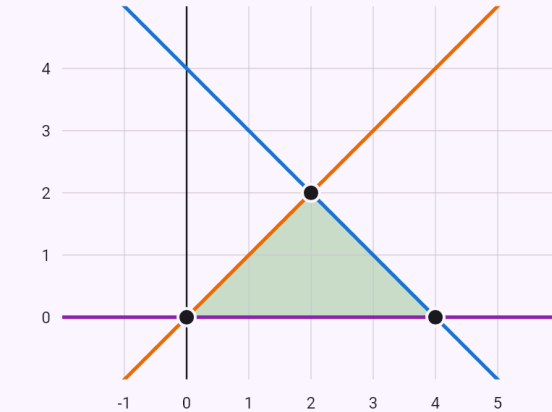
$$y \in [0, \infty)$$

Steps

- Inequality 1: $x + y \leq 4$
- The boundary is the curve $x + y = 4$ – drawn solid, its points belong to the solution.
- Substitute the point $[0; 0]$: $0 \leq 4$ holds – the solution is the part of the plane containing the point.
- Inequality 2: $x - y \geq 0$
- The boundary is the curve $x - y = 0$ – drawn solid, its points belong to the solution.
- Substitute the point $[1; 0]$: $1 \geq 0$ holds – the solution is the part of the plane containing the point.
- Inequality 3: $y \geq 0$
- The boundary is the curve $y = 0$ – drawn solid, its points belong to the solution.
- Substitute the point $[0; 1]$: $1 \geq 0$ holds – the solution is the part of the plane containing the point.
- The system's solution is the common part of these regions – shaded in the graph.

- The region's corners (where the boundaries meet): $[0; 0]$, $[2; 2]$, $[4; 0]$

Graph



$$\begin{aligned} \text{blue } x + y \leq 4 &\Leftrightarrow y \leq 4 - x \\ \text{orange } x - y \geq 0 &\Leftrightarrow y \leq x \\ \text{purple } y \geq 0 &\Leftrightarrow y \in [0, \infty) \end{aligned}$$

The solution is the shaded region. A solid boundary belongs to it, a dashed one doesn't.

The region's corners:

$$\bullet (0, 0) \quad \bullet (2, 2) \quad \bullet (4, 0)$$

An inequality in the integers – the integers of the interval are listed.

Input: $x^2 < 10$

Result

$$x \in \{-3, -2, -1, 0, 1, 2, 3\}$$

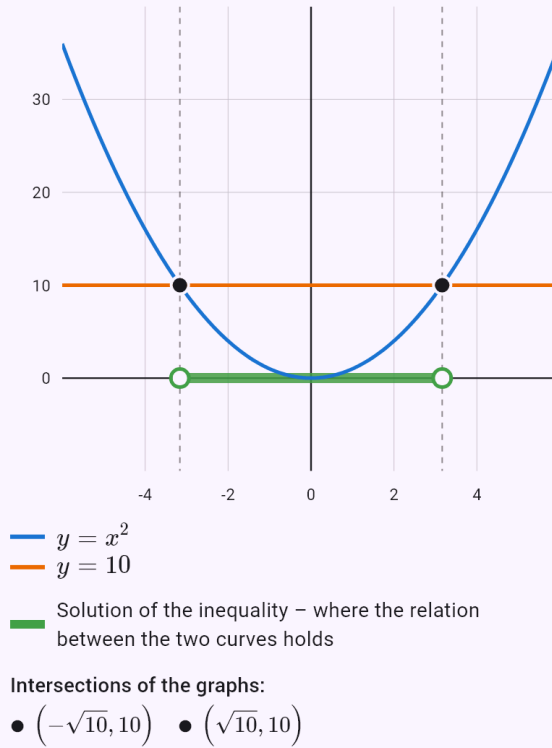
Steps

- Move everything to one side: $x^2 - 10 < 0$
- The critical points split the number line into intervals:
 $x_1 = -\sqrt{10}$, $x_2 = \sqrt{10}$
- Signs on the individual intervals:

$x^2 - 10$	$(-\infty, -\sqrt{10})$	$(-\sqrt{10}, \sqrt{10})$	$(\sqrt{10}, \infty)$
	+	-	+

- In the real numbers $x \in (-\sqrt{10}, \sqrt{10})$; in $\mathbb{Z}$:
 $x \in \{-3, -2, -1, 0, 1, 2, 3\}$

Graph



An equation and an inequality – the equation's roots substituted into the inequality.

Input: $x^2 - x - 6 = 0$

$x > 0$

Result

$$x = 3$$

Steps

- First we solve the equation $x^2 - x - 6 = 0$, then substitute its roots into the inequalities.
- Compute the discriminant:
 $D = b^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot (-6) = 25$
- Substitute into the quadratic formula:
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 5}{2}$$
- The root $x = -2$ does not satisfy the condition $x > 0$, so it is not a solution.
- The root $x = 3$ satisfies the conditions.

A circle and a half-plane – the solution is an arc, its end points don't belong to it.

Input: $x^2 + y^2 = 4$

$$y > x$$

Result

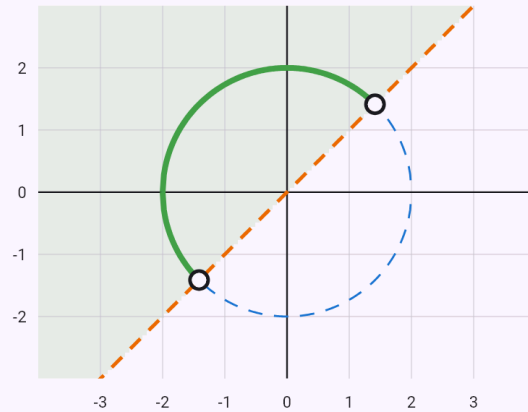
$$x^2 + y^2 = 4$$

$$y > x$$

Steps

- The solution is the points of the curve $x^2 + y^2 = 4$ lying in the region given by the inequalities. First we find this region:
- The boundary is the curve $y = x$ – drawn dashed, its points don't belong to the solution.
- Substitute the point $[1; 0]$: $0 > 1$ doesn't hold – the solution is the part of the plane not containing the point.
- The end points of the curve's part: $[-\sqrt{2}; -\sqrt{2}]$, $[\sqrt{2}; \sqrt{2}]$ – on a dashed boundary, not part of the solution.

Graph



$$\text{---} x^2 + y^2 = 4 \Leftrightarrow y = \pm\sqrt{4 - x^2}$$

$$\text{---} y > x$$

The solution is the part of the curve highlighted in green – the part lying in the shaded region. A filled end point belongs to the solution, a hollow one doesn't.

End points:

$$\circ (-\sqrt{2}, -\sqrt{2}) \quad \circ (\sqrt{2}, \sqrt{2})$$

Known limitations

- A system with two or more equations together with inequalities is not solved yet; write decimals with a point ($x > 2.5$).

Systems of equations

The app solves a system of linear equations in two unknowns by elimination (addition) or substitution, and larger systems by Gaussian elimination – also in matrix form. You choose the method with a chip. A system of two equations in two unknowns also gets a graph in which the solutions are the intersections of the lines or curves.

A nonlinear system (say a circle and a line) is solved and drawn, so far without the working.

Examples

The elimination (addition) method.

Input: $2x + 3y = 7$

$x - y = 1$

Addition

Result

$$x = 2$$

$$y = 1$$

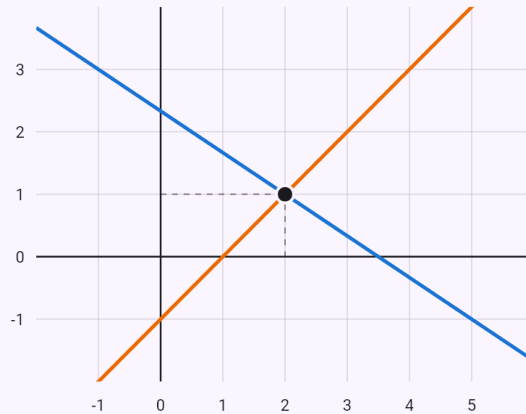
Steps

- Multiply the first equation by 1 and the second by -2 so that x cancels when they are added

$$\begin{cases} 2x + 3y = 7 \\ -2x + 2y = -2 \end{cases}$$

- Add the equations, x cancels: $5y = 5$
- Solve: $5y = 5 \Rightarrow y = 1$
- Substitute into equation 2: $x - 1 = 1 \Rightarrow x = 2$

Graph



$$\text{— } 2x + 3y = 7 \Leftrightarrow y = \frac{7}{3} - \frac{2x}{3}$$

$$\text{— } x - y = 1 \Leftrightarrow y = x - 1$$

Intersections of the graphs:

- $(2, 1)$

Each intersection corresponds to one solution of the system – its coordinates are the values of x and y .

Three equations – Gaussian elimination.

Input: $x + y + z = 6$

$2x - y + z = 3$

$$x + 2y - z = 2$$

Gaussian elimination

Result

$$x = 1$$

$$y = 2$$

$$z = 3$$

Steps

- Eliminate x from the equations below

$$\begin{cases} x + y + z = 6 \\ -3y - z = -9 \\ y - 2z = -4 \end{cases}$$

- Eliminate y from the equations below

$$\begin{cases} x + y + z = 6 \\ -3y - z = -9 \\ -z = -3 \end{cases}$$

- Solve: $-z = -3 \Rightarrow z = 3$
- Substitute into equation 2: $-3y - 3 = -9 \Rightarrow y = 2$
- Substitute into equation 1: $x + 5 = 6 \Rightarrow x = 1$

A circle and a line.

Input: $x^2 + y^2 = 25$

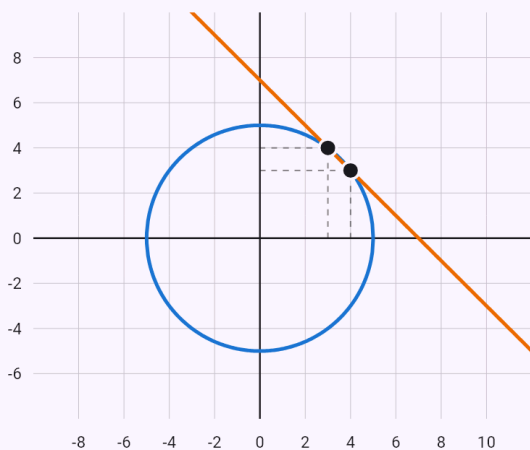
$$x + y = 7$$

Result

$$x_1 = 3, y_1 = 4$$

$$x_2 = 4, y_2 = 3$$

Graph



$$\text{— } x^2 + y^2 = 25 \Leftrightarrow y = \pm \sqrt{25 - x^2}$$

$$\text{— } x + y = 7 \Leftrightarrow y = 7 - x$$

Intersections of the graphs:

$$\bullet (3, 4) \quad \bullet (4, 3)$$

Each intersection corresponds to one solution of the system – its coordinates are the values of x and y .

Known limitations

- Nonlinear systems come without the working for now.

Equations and inequalities with a parameter

A letter in an equation besides the unknown is a parameter – any real number the solution depends on. The app rearranges the equation into the form $A \cdot x = B$ (a quadratic one into $A \cdot x^2 + B \cdot x + C = 0$) and discusses the solution by the parameter's values: where the coefficient of the unknown is zero, where the discriminant is positive, zero or negative, where the conditions exclude a root and for which values the problem makes no sense. The result sums up the cases.

The app takes the unknown in the order x, y, z, u, v, w, t ; the other letters are parameters. Choose another unknown with the “Unknown” chip or write it after a comma. The graph shows all solutions as a curve (a region for an inequality) in the parameter–unknown plane; pick the parameter's value with the slider and read the solutions off the vertical line.

In an indefinite integral, other letters are constants – the app adds which must not be zero ($\int dx/(a - bx)^3$ for $b \neq 0$).

How to enter it

$$2ax + a = 4x$$

Every other letter is a parameter; the unknown is x (or $y, z, \dots$).

$$(q+3)x = 2x + q, \text{ unknown } q$$

You can name the unknown or the parameter after a comma (“ a is a parameter”, “unknown q ”).

Examples

A linear equation – no solution for $a = 2$.

Input: $2ax + a = 4x$

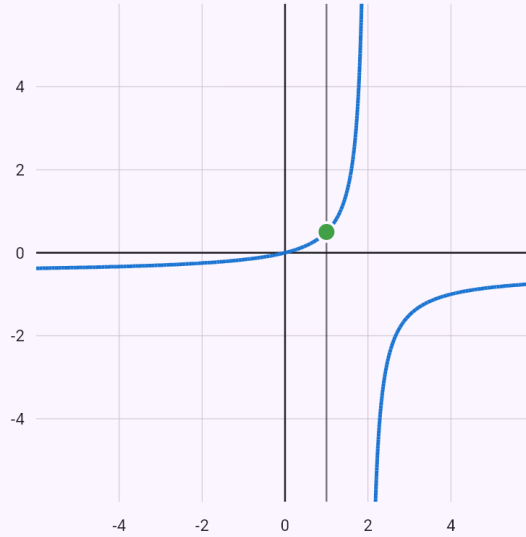
Result

$$\begin{cases} x = \frac{a}{2(2-a)}, & a \neq 2 \\ x \in \emptyset, & a = 2 \end{cases}$$

Steps

- The unknown is x , the parameter a is any real number.
The solution depends on its value, so we go through the individual cases (discussion of the solution).
- Rearrange into the basic form: $2(a-2)x = -a$
- For $a \neq 2$, $2(a-2)$ is non-zero, so we can divide by it:
$$x = \frac{a}{2(2-a)}$$
- For $a = 2$, the coefficient of the unknown is zero:
 $0 \cdot x = -2$, so $x \in \emptyset$

Graph



— $2xa + a = 4x$

Every point $[a; x]$ of the curve is a solution. For a chosen value of a , the solutions lie on the vertical line – move it with the slider.



$a = 1 : x \approx 0.5$

A quadratic equation – discussed by the discriminant.

Input: $x^2 + 6x + p = 0$

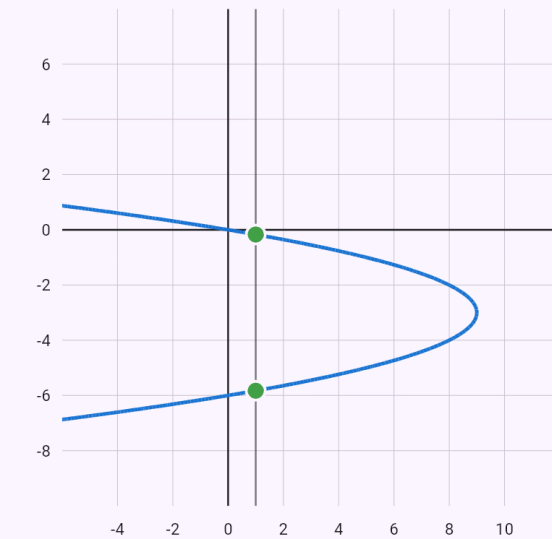
Result

$$\begin{cases} x_{1,2} = -3 \pm \sqrt{9 - p}, & p < 9 \\ x = -3, & p = 9 \\ x \in \emptyset, & p > 9 \end{cases}$$

Steps

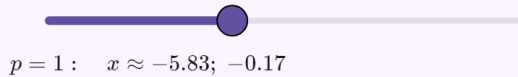
- The unknown is x , the parameter p is any real number. The solution depends on its value, so we go through the individual cases (discussion of the solution).
- Rearrange into the basic form: $x^2 + 6x + p = 0$
- The discriminant depending on the parameter:
 $D = 6^2 - 4 \cdot 1 \cdot p = 36 - 4p = 4(9 - p)$
- For $p < 9$, $D > 0$ and the equation has two different roots:
 $x_{1,2} = -3 \pm \sqrt{9 - p}$
- For $p > 9$, $D < 0$ and the equation has no real solution.
- For $p = 9$, $D = 0$ and the equation has one double root:
 $x = -3$

Graph



$$x^2 + 6x + p = 0$$

Every point $[p; x]$ of the curve is a solution. For a chosen value of p , the solutions lie on the vertical line – move it with the slider.



$p = 1 : x \approx -5.83; -0.17$

An inequality – the coefficient's sign decides the direction.

Input: $2x + 2p - 5 < 2px + 7$

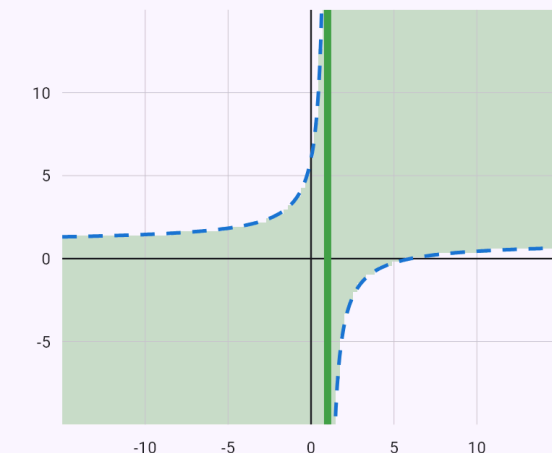
Result

$$\begin{cases} x < \frac{p-6}{p-1}, & p < 1 \\ x \in \mathbb{R}, & p = 1 \\ x > \frac{p-6}{p-1}, & p > 1 \end{cases}$$

Steps

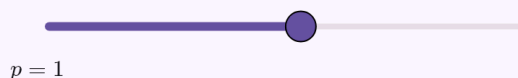
- The unknown is x , the parameter p is any real number. The solution depends on its value, so we go through the individual cases (discussion of the solution).
- Rearrange into the basic form: $2(1 - p)x < 2(6 - p)$
- For $p < 1$, $2(1 - p)$ is positive – dividing keeps the inequality sign: $x < \frac{p-6}{p-1}$
- For $p > 1$, $2(1 - p)$ is negative – dividing reverses the inequality sign: $x > \frac{p-6}{p-1}$
- For $p = 1$, the coefficient of the unknown is zero:
 $0 \cdot x < 10$, so $x \in \mathbb{R}$

Graph



$$2x + 2p - 5 < 2px + 7 \Leftrightarrow x = \frac{p-6}{p-1}$$

The shaded region contains all solutions $[p; x]$. For a chosen value of p , the solution is the highlighted part of the vertical line – move it with the slider.



$p = 1$

Known limitations

- So far linear and quadratic equations and linear inequalities; the full discussion by intervals with one parameter only, with more parameters the general cases.
- Questions such as “for which p does the equation have a positive root” and systems with a parameter are not supported yet.

Simplifying expressions

The app transforms an expression without an equals sign according to the chosen chip: it simplifies it (a rational expression over a common denominator and cancelled, with conditions), expands it, factors it (by taking out a common factor, by formulas and by synthetic division) or splits it into partial fractions – as on paper, with equations for the unknown coefficients. Factoring is over the integers; if the expression factors further over the reals (by irrational roots), the next step shows it. A root of a monomial is taken out partly – with an even root as an absolute value where the base may be negative. Complex numbers work too – a quotient by multiplying with the complex conjugate.

Examples

Simplifying a rational expression.

Input: $(x^2 - 1)/(x^2 + 2x + 1)$

Simplify

Result

$$\frac{x - 1}{x + 1}$$

Conditions: $x \neq -1$

Steps

- Conditions – a denominator must not be zero: $x \neq -1$
- Factor the numerator and denominator: $\frac{(x - 1)(x + 1)}{(x + 1)^2}$
- Cancel the common factor $x + 1$: $\frac{x - 1}{x + 1}$

Expanding.

Input: $(x + 2)^3$

Expand

Result

$$x^3 + 6x^2 + 12x + 8$$

Steps

- Multiply out the brackets and collect like terms:
 $x^3 + 6x^2 + 12x + 8$

Factoring by synthetic division.

Input: $x^3 - 6x^2 + 11x - 6$ Factor

Result

$$(x - 3)(x - 2)(x - 1)$$

Steps

- Rational roots can only be divisors of the constant term -6 : $\pm 1, \pm 2, \pm 3, \pm 6$

- Horner's scheme confirms the root $x = 1$ and divides the polynomial by it:

$$\begin{array}{r|rrrr} & 1 & -6 & 11 & -6 \\ 1 & 1 & -5 & 6 & 0 \end{array}$$

$$P(x) = (x - 1)(x^2 - 5x + 6)$$

- Substituting shows these candidates aren't roots:

$$P(-1) = -24 \neq 0$$

- Horner's scheme confirms the root $x = 2$ and divides the polynomial by it:

$$\begin{array}{r|rrrr} & 1 & -5 & 6 \\ 2 & 1 & -3 & 0 \end{array}$$

$$P(x) = (x - 1)(x - 2)(x - 3)$$

Partial fractions.

Input: $(3x + 5)/(x^2 + x - 2)$ Partial fractions

Result

$$\frac{8}{3(x - 1)} + \frac{1}{3(x + 2)}$$

Conditions: $x \neq -2, x \neq 1$

Steps

- Conditions – a denominator must not be zero:

$$x \neq -2, x \neq 1$$

- Factor the denominator: $(x - 1)(x + 2)$

- Look for a decomposition of the form: $\frac{A}{x - 1} + \frac{B}{x + 2}$

- Multiply both sides by the denominator $(x - 1)(x + 2)$:

$$3x + 5 = A(x + 2) + B(x - 1)$$

- Substitute $x = 1$: $8 = 3A \Rightarrow A = \frac{8}{3}$

- Substitute $x = -2$: $-1 = -3B \Rightarrow B = \frac{1}{3}$

Taking a root out partly, with an absolute value.

Input: $\sqrt{8a^5 b^2}$ [Simplify](#)

Result

$$2a^2|b|\sqrt{2a}$$

Steps

- Take what we can out of the root – powers under it come out in front (with an even root as an absolute value where the base may be negative): $2a^2|b|\sqrt{2a}$

Dividing complex numbers.

Input: $(3+4i)/(1-2i)$ [Simplify](#)

Result

$$-1 + 2i$$

Steps

- Multiply the fraction by the denominator's complex conjugate, $1 + 2i$

$$\frac{(3+4i)(1+2i)}{(1-2i)(1+2i)} = \frac{-5+10i}{5} = -1+2i$$

Limits

MathSolver computes the limit of a function of one variable at a point, at infinity, and one-sided limits, and proceeds as at school: first it tries to substitute into the function. The limit of a sequence ($n \rightarrow \infty$) works the same way – the form 1^∞ too, or a sum of powers with the fastest growing term taken out.

If substitution gives an indeterminate form ($0/0$, ∞/∞ , ...), it chooses a transformation by the form of the function – factoring and cancelling, multiplying by the conjugate, dividing by the highest power, a known limit ($\sin x / x$) or L'Hôpital's rule, repeatedly if needed.

An infinite limit, or a limit that does not exist (the one-sided limits differ), is never passed off as a number – the app says so in words and shows the one-sided limits. The result comes with a graph of the function near the point with the limit value marked.

For the limit of a sequence the graph shows the points $a_1, a_2, a_3, \dots$ instead of a continuous curve.

How to enter it

$\lim_{x \rightarrow 2} (x^2 - 4) / (x - 2)$	A typed limit: after the word lim, in brackets the variable, the arrow $\rightarrow$ and the point, then the function.
$\lim_{x \rightarrow \infty} (3x^2 + 1) / (x^2 - 5)$	A limit at infinity: infinity is written ∞ (minus infinity - ∞).
$\lim_{x \rightarrow 0^+} 1/x$	A one-sided limit: + (from the right) or - (from the left) after the point.
$\lim_{x \rightarrow 0} \frac{\sin x}{x}$	From a photo, or in LaTeX as the camera recognizes it.
$\lim_{n \rightarrow \infty} ((n+2)/(n+1))^{(2n+3)}$	The limit of a sequence at infinity.

Examples

The indeterminate form 0/0 – we factor the numerator and cancel.

Input: $\lim_{x \rightarrow 2} (x^2 - 4)/(x - 2)$

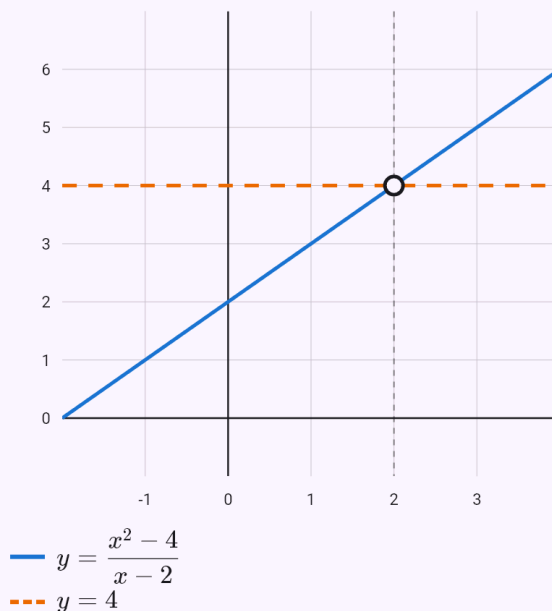
Result

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$$

Steps

- Substituting gives the indeterminate form $\frac{0}{0}$ – the expression has to be rewritten.
- Rewrite and cancel
$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2)$$
- Substitute: $\lim_{x \rightarrow 2} (x + 2) = 2 + 2 = 4$

Graph



Limit:

● (2, 4)

Hollow dot – the value the function approaches (it need not be defined at the point itself); dashed – the limit.

A limit at infinity – numerator and denominator divided by the highest power.

Input: $\lim(x \rightarrow \infty) (3x^2+1)/(x^2-5)$

Result

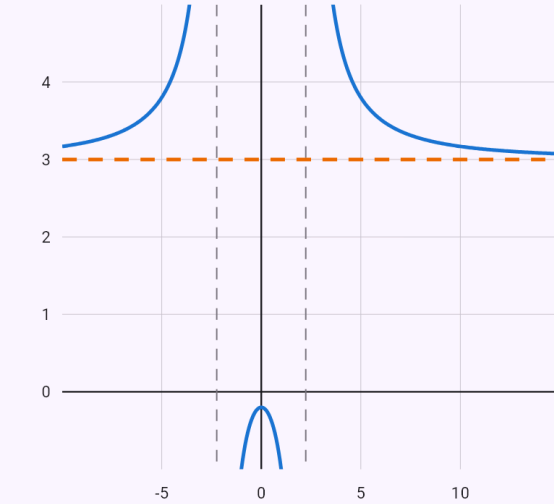
$$\lim_{x \rightarrow \infty} \frac{3x^2 + 1}{x^2 - 5} = 3$$

Steps

- Substituting gives the indeterminate form $\frac{\infty}{\infty}$ – the expression has to be rewritten.
- Divide the numerator and the denominator by x^2 – terms like c/x^n tend to zero

$$\lim_{x \rightarrow \infty} \frac{3x^2 + 1}{x^2 - 5} = \lim_{x \rightarrow \infty} \frac{3 + \frac{1}{x^2}}{1 - \frac{5}{x^2}} = 3$$

Graph



$$\begin{aligned} \text{— } y &= \frac{3x^2 + 1}{x^2 - 5} \\ \text{--- } y &= 3 \end{aligned}$$

Hollow dot – the value the function approaches (it need not be defined at the point itself); dashed – the limit.

The known limit $\sin x / x$.

Input: $\lim(x \rightarrow 0) \sin(3x)/x$

Result

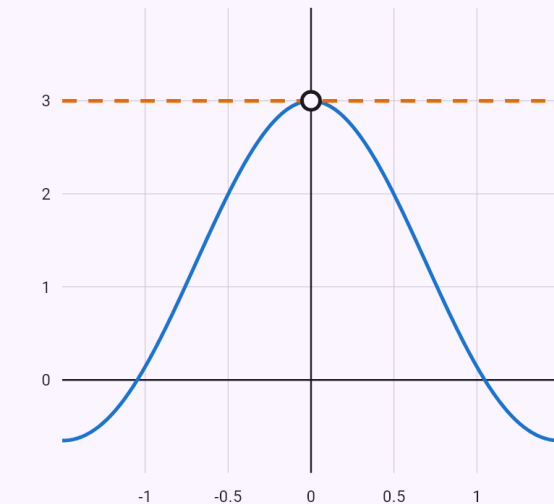
$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = 3$$

Steps

- Substituting gives the indeterminate form $\frac{0}{0}$ – the expression has to be rewritten.
- Use the known limit $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} \cdot 3 = 1 \cdot 3 = 3$$

Graph



$$\begin{aligned} \text{— } y &= \frac{\sin(3x)}{x} \\ \text{--- } y &= 3 \end{aligned}$$

Limit:

• $(0, 3)$

Hollow dot – the value the function approaches (it need not be defined at the point itself); dashed – the limit.

L'Hôpital's rule, applied twice in a row.

Input: $\lim_{x \rightarrow 0} (e^x - 1 - x) / x^2$

Result

$$\lim_{x \rightarrow 0} \frac{-x + e^x - 1}{x^2} = \frac{1}{2}$$

Steps

- Substituting gives the indeterminate form $\frac{0}{0}$ – the expression has to be rewritten.

- L'Hôpital's rule – differentiate the numerator and the denominator

$$\lim_{x \rightarrow 0} \frac{-x + e^x - 1}{x^2} = \lim_{x \rightarrow 0} \frac{(-x + e^x - 1)'}{(x^2)'} = \lim_{x \rightarrow 0} \frac{e^x - 1}{2x}$$

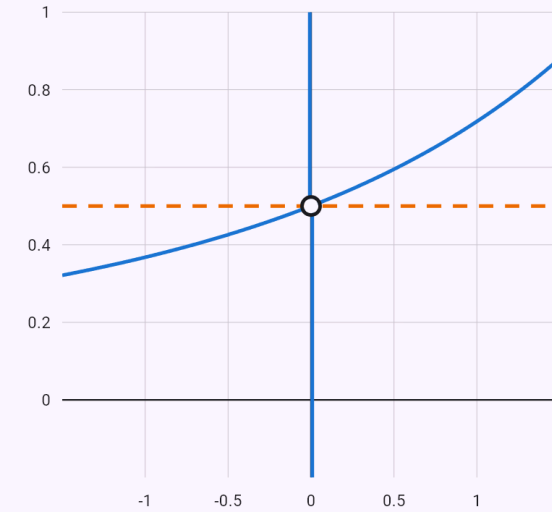
- Substituting gives the indeterminate form $\frac{0}{0}$ – the expression has to be rewritten.

- L'Hôpital's rule – differentiate the numerator and the denominator

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{2x} = \lim_{x \rightarrow 0} \frac{(e^x - 1)'}{(2x)'} = \lim_{x \rightarrow 0} \frac{e^x}{2}$$

- Substitute: $\lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$

Graph



$$y = \frac{e^x - 1 - x}{x^2}$$

$$y = \frac{1}{2}$$

Limit:

$$\bullet \left(0, \frac{1}{2}\right)$$

Hollow dot – the value the function approaches (it need not be defined at the point itself); dashed – the limit.

The one-sided limits differ – the limit does not exist.

Input: $\lim(x \rightarrow 0) \ 1/x$

Result

The limit does not exist – the one-sided limits differ.

Steps

- Substituting gives a non-zero number over zero – the limit is infinite, its sign from the left and from the right:

$$\lim_{x \rightarrow 0} \frac{1}{x} \rightarrow \frac{1}{0}$$

- One-sided limits: $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$, $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$

Graph



$$y = \frac{1}{x}$$

The limit of a sequence – take out the fastest growing term 9^n .

Input: $\lim_{n \rightarrow \infty} (6^n + 7^n + 9^n)^{\frac{1}{2n}}$

Result

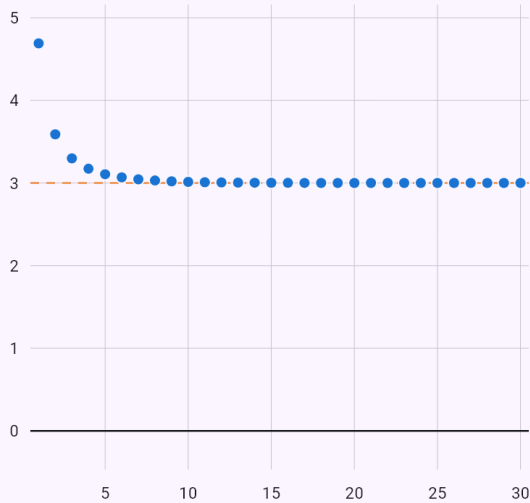
$$\lim_{n \rightarrow \infty} (6^n + 7^n + 9^n)^{\frac{1}{2n}} = 3$$

Steps

- Take out the fastest growing term 9^n – the others divided by it tend to zero

$$\lim_{n \rightarrow \infty} (6^n + 7^n + 9^n)^{\frac{1}{2n}} = \lim_{n \rightarrow \infty} \sqrt[2n]{9^n \cdot \left(\frac{6^n}{9^n} + \frac{7^n}{9^n} + 1 \right)} =$$

Graph



● terms a_n — limit of the sequence

The points are the terms a_n for $n = 1, 2, 3, \dots$. If the sequence has a limit, the terms approach it (dashed).

Derivatives

The app differentiates by the rules – the sum, constant multiple, product, quotient and chain rules – and says at every step which rule it used. It also handles higher derivatives and the value of the derivative at a point. Basic functions (x^2 , $\sin x$, e^x , ...) can also be differentiated from the definition, as the limit of the difference quotient. The result comes with a graph of the function and its derivative.

For a function given implicitly by an equation in x and y ($x^3 + y^3 = 3xy$) it differentiates both sides by x (y as a function of x) and expresses y' , or uses the formula $y' = -F_x / F_y$; it also finds y'' and, at a given point, the tangent and the normal, and draws the curve.

At a singular point (the curve crosses itself or has a cusp) it finds the tangents from the equation's lowest-degree terms – for the folium at the origin $x = 0$ and $y = 0$.

How to enter it

$(x^2 \sin x)'$	A derivative with a prime after the bracket.
$d/dx (e^x / x)$	The d/dx notation, also from a photo ($\frac{d}{dx}$).
$(x^3 \ln x)''$	Higher derivatives with more primes, or with the order at the “Differentiate” chip.
$(x^2 + 1)' _{x=2}$	The derivative at a point, or fill in the point at the chip.
$x^2 \sin x$	An expression with the “Differentiate” chip – there you also choose the order, the point and the method (by the rules / from the definition).
$x^3 + y^3 = 3xy$	An equation in x and y – the “Implicit derivative” chip (order 1 or 2, the method, a point $[x_0; y_0]$) or “Graph of the curve”; written $\text{diff}(x^3+y^3=3xy, x)$.

Examples

The product rule.

Input: $(x^2 \sin x)'$

Result

$$y' = 2x \sin x + x^2 \cos x$$

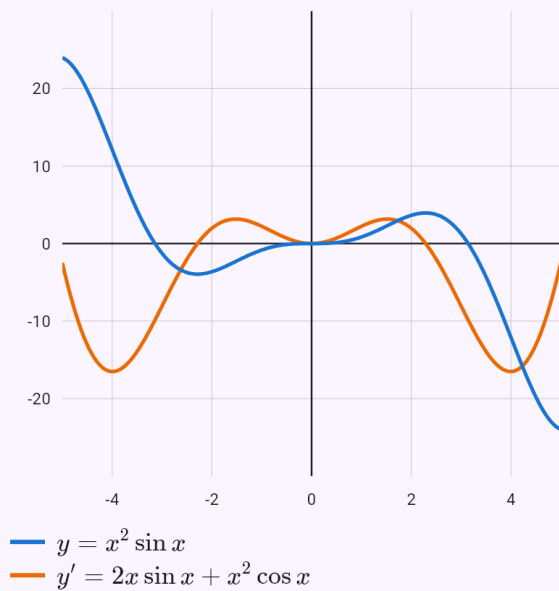
Steps

- Product rule $(u \cdot v)' = u' \cdot v + u \cdot v'$

$$(x^2 \sin x)' = (x^2)' \cdot \sin x + x^2 \cdot (\sin x)' = 2x \sin x + x^2 \cos x$$

Where the derivative is positive, the function increases; where it is negative, it decreases. At the derivative's zeros the function has a horizontal tangent.

Graph



The quotient rule.

Input: $\frac{d}{dx} (e^x / x)$

Result

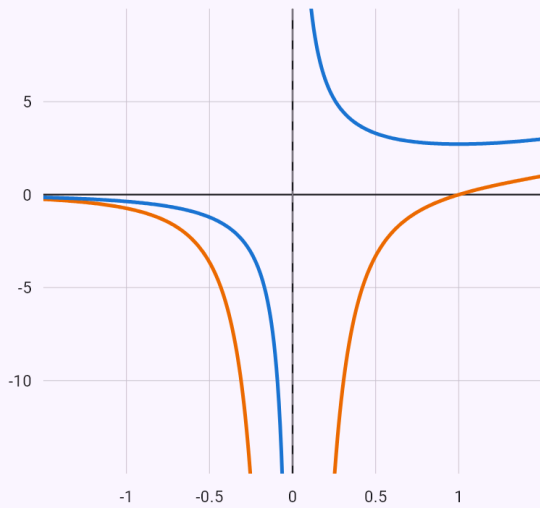
$$y' = \frac{xe^x - e^x}{x^2}$$

Steps

- Quotient rule $(u/v)' = (u' \cdot v - u \cdot v')/v^2$

$$\left(\frac{e^x}{x}\right)' = \frac{(e^x)' \cdot x - e^x \cdot (x)'}{x^2} = \frac{xe^x - e^x}{x^2}$$

Graph



$$y = \frac{e^x}{x}$$
$$y' = \frac{xe^x - e^x}{x^2}$$

Where the derivative is positive, the function increases; where it is negative, it decreases. At the derivative's zeros the function has a horizontal tangent.

The second derivative.

Input: $(x^3 \ln x)''$

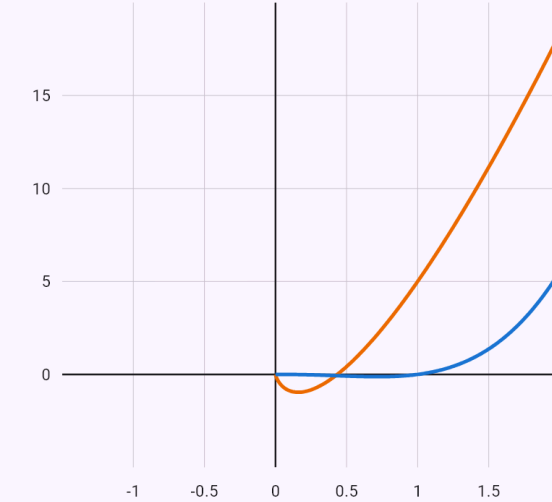
Result

$$y'' = 5x + 6x \ln x$$

Steps

- Derivative number 1: $y' = (x^3 \ln x)'$
- Product rule $(u \cdot v)' = u' \cdot v + u \cdot v'$
 $(x^3 \ln x)' = (x^3)' \cdot \ln x + x^3 \cdot (\ln x)' = 3x^2 \ln x + x^2$
- Derivative number 2: $y'' = (x^2 + 3x^2 \ln x)'$
- The derivative of a sum is the sum of the derivatives:
 $(x^2 + 3x^2 \ln x)' = (x^2)' + (3x^2 \ln x)'$
- Take the constant out of the derivative:
 $(3x^2 \ln x)' = 3 \cdot (x^2 \ln x)'$
- Product rule $(u \cdot v)' = u' \cdot v + u \cdot v'$
 $(x^2 \ln x)' = (x^2)' \cdot \ln x + x^2 \cdot (\ln x)' = 2x \ln x + x$
- Substitute:
 $(3x^2 \ln x)' = 3 \cdot (x + 2x \ln x) = 3(x + 2x \ln x)$
- Substitute: $(x^2 + 3x^2 \ln x)' = 2x + 3(x + 2x \ln x)$
- Simplify: $y'' = 5x + 6x \ln x$

Graph



$y = x^3 \ln x$
 $y'' = 5x + 6x \ln x$

Where the second derivative is positive, the function is convex (bends upwards); where it is negative, it is concave (bends downwards).

The derivative from the definition, as a limit.

Input: x^2 Differentiate → From the definition

Result

$$y' = 2x$$

Steps

- The derivative from the definition

$$y' = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$
- Substituting gives the indeterminate form $\frac{0}{0}$ - the expression has to be rewritten.
- Rewrite and cancel

$$\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x+h)}{h} = \lim_{h \rightarrow 0} (2x+h)$$
- Substitute: $\lim_{h \rightarrow 0} (2x+h) = 2x$

Graph



$y = x^2$
 $y' = 2x$

Where the derivative is positive, the function increases; where it is negative, it decreases. At the derivative's zeros the function has a horizontal tangent.

An implicit derivative – the folium of Descartes, the tangent and the normal at $[3/2; 3/2]$, the second derivative.

Input: $x^3 + y^3 = 3xy$ Implicit derivative, order 2, point $[3/2; 3/2]$

Result

$$y' = -\frac{x^2 - y}{y^2 - x}$$

$$y'' = -\frac{2xy}{(y^2 - x)^3}$$

$$y'(\frac{3}{2}) = -1$$

$$t: y = 3 - x$$

$$n: y = x$$

$$y''(\frac{3}{2}) = -\frac{32}{3}$$

Conditions: $y^2 - x \neq 0$

Steps

- The function $y = y(x)$ is given by the equation $x^3 + y^3 = 3xy$ – we differentiate both sides by x , taking y as a function of x (the chain rule: e.g. $(y^3)' = 3y^2 \cdot y'$).
- Differentiating both sides:
 $(x^3 + y^3)' = (3xy)' \Rightarrow 3y^2 y' + 3x^2 = 3xy' + 3y$
- The terms with y' to the left, the rest to the right, y' factored out: $y'(3y^2 - 3x) = -3x^2 + 3y$

- Expressing y' : $y' = -\frac{x^2 - y}{y^2 - x}$

- The second derivative – y' differentiated by x again (y and y' are functions of x)

$$y'' = \left(-\frac{x^2 - y}{y^2 - x} \right)' = \frac{2x^2 y y' - 2xy^2 - y^2 y' + x^2 - xy'}{y^4 - 2xy^2 + x^2}$$

- Substituting y'

$$y'' = \frac{2xy(x + y + 1)(x^2 - xy + y^2 - x - y + 1)}{(-y^2 + x)^3}$$

- On the curve $x^3 + y^3 - 3xy = 0$ holds – which simplifies it: $y'' = -\frac{2xy}{(y^2 - x)^3}$

- The point $[\frac{3}{2}; \frac{3}{2}]$ lies on the curve:
 $x^3 + y^3 - 3xy|_{x=\frac{3}{2}, y=\frac{3}{2}} = 0$

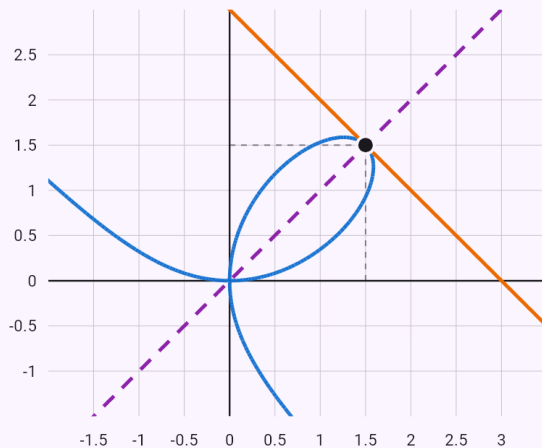
- The derivative at the point $[\frac{3}{2}; \frac{3}{2}]$ – the tangent's slope:
 $y'(\frac{3}{2}) = -1$

- The tangent: $y - \frac{3}{2} = -\left(x - \frac{3}{2}\right) \Rightarrow y = 3 - x$

- The normal is perpendicular to the tangent, its slope is 1:
 $y - \frac{3}{2} = \left(x - \frac{3}{2}\right) \Rightarrow y = x$

- The second derivative at the point: $y''(\frac{3}{2}) = -\frac{32}{3}$

Graph



— $x^3 + y^3 = 3xy$
— t: $y = 3 - x$
- - - n: $y = x$

The curve given by the equation. At the given point, the tangent (solid) and the normal (dashed).

Function analysis

The app analyzes a function in the usual order: domain, even or odd, intercepts, the sign of the function, limits at the ends of the domain and asymptotes, the first derivative – monotonicity and extrema (also by higher derivatives), the second derivative – concavity and inflection points, and finally the range. It sums everything up in an overview and draws a graph with the points marked.

For a trigonometric function the app finds the period, examines the function on one period and writes the results with the period (a maximum at $x = \pi/2 + 2k\pi$); if only the derivative is periodic ($x + 2 \sin x$), the extrema and monotonicity repeat with the period. Absolute values are removed by the signs on the intervals and the one-sided derivatives compared at the corners; where the derivative is infinite, a vertical tangent is shown. Nested absolute values and sums of absolute values are analyzed by pieces; where the function is constant on an interval, this is stated with its value (a minimum at every point of the interval).

How to enter it

analysis $y = x^3 - 3x$

With the word “analysis” before the function.

$x^3 - 3x$

A function with the “Analyze function” chip.

Examples

A polynomial – extrema and an inflection point.

Input: $x^3 - 3x$ Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = \mathbb{R}$

Parity: odd

x-intercepts: $[-\sqrt{3}; 0]$, $[0; 0]$, $[\sqrt{3}; 0]$

Increasing: $(-\infty, -1]$, $[1, \infty)$

Decreasing: $[-1, 1]$

Local maximum: $[-1; 2]$

Local minimum: $[1; -2]$

Convex: $[0, \infty)$

Concave: $(-\infty, 0]$

Points of inflection: $[0; 0]$

Steps

- Domain: $D(f) = \mathbb{R}$
- $f(-x) = (-x)^3 - 3(-x) = -x^3 + 3x = -f(x)$ - the function is odd (its graph is symmetric about the origin).
- Intersections with the x-axis:
 $f(x) = 0 \Rightarrow x = -\sqrt{3}, x = 0, x = \sqrt{3}$, i.e.
 $[-\sqrt{3}; 0]$, $[0; 0]$, $[\sqrt{3}; 0]$
- Intersection with the y-axis: $f(0) = 0$, i.e. $[0; 0]$
- The sign of the function between its zeros and the points outside the domain:

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
$f(x)$	-	+	-	+

The function is positive for $x \in (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$
and negative for $x \in (-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

- The limits at the ends of the domain

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

- The function has no vertical asymptotes.
- No asymptote at $\pm\infty$: $k = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \infty$
- The first derivative: $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$
- Stationary points: $f'(x) = 0 \Rightarrow x = -1, x = 1$
- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$f'(x)$	+	-	+
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

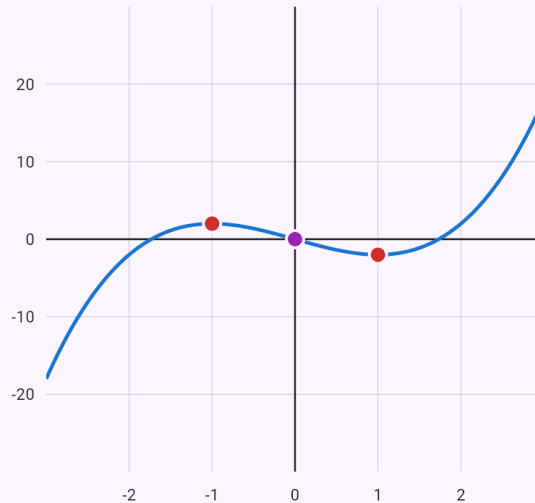
The function is increasing on $(-\infty, -1]$, $[1, \infty)$ and decreasing on $[-1, 1]$

- At $x = -1$: $f''(-1) = -6 < 0$ - a local maximum.
- At $x = 1$: $f''(1) = 6 > 0$ - a local minimum.
- Local maximum $[-1; 2]$, local minimum $[1; -2]$

- The second derivative: $f''(x) = 6x$
- Candidates for points of inflection: $f''(x) = 0 \Rightarrow x = 0$
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(-\infty, 0)$	$(0, \infty)$
$f''(x)$	-	+
$f(x)$	∩	∪
- The function is convex on $[0, \infty)$ and concave on $(-\infty, 0]$
- At $x = 0$: $f'''(0) = 6 > 0$ – the first non-zero derivative (from the third on) is of an odd order: a point of inflection.
- Points of inflection: $[0; 0]$
- Range – from the monotonicity, extrema and limits:
 $R(f) = \mathbb{R}$

Graph



$y = x^3 - 3x$

Notable points:

- maximum $(-1, 2)$ • inflection $(0, 0)$
- minimum $(1, -2)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

A rational function – asymptotes.

Input: $x^2/(x - 1)$ [Analyze function](#)

Result

Domain: $D(f) = \mathbb{R} \setminus \{1\}$
 Range: $R(f) = (-\infty, 0] \cup [4, \infty)$
 Parity: neither even nor odd
 x-intercepts: $[0; 0]$
 Increasing: $(-\infty, 0]$, $[2, \infty)$
 Decreasing: $[0, 1)$, $(1, 2]$
 Local maximum: $[0; 0]$
 Local minimum: $[2; 4]$
 Convex: $(1, \infty)$
 Concave: $(-\infty, 1)$
 Asymptotes: $x = 1$, $y = x + 1$

Steps

- Domain – conditions $x \neq 1$: $D(f) = \mathbb{R} \setminus \{1\}$
- The domain $D(f) = \mathbb{R} \setminus \{1\}$ isn't symmetric about zero – the function is neither even nor odd.
- Intersections with the x-axis: $f(x) = 0 \Rightarrow x = 0$, i.e. $[0; 0]$
- Intersection with the y-axis: $f(0) = 0$, i.e. $[0; 0]$
- The sign of the function between its zeros and the points outside the domain:

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f(x)$	-	-	+

The function is positive for $x \in (1, \infty)$ and negative for $x \in (-\infty, 0) \cup (0, 1)$

- The limits at the ends of the domain

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

$$\lim_{x \rightarrow 1^-} f(x) = -\infty$$

$$\lim_{x \rightarrow 1^+} f(x) = \infty$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

- Vertical asymptotes (an infinite limit there): $x = 1$

- Asymptote at $\pm\infty$: $k = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = 1$,
 $q = \lim_{x \rightarrow \pm\infty} (f(x) - x) = 1$ – the line $y = x + 1$

- The first derivative

$$f'(x) = \frac{x^2 - 2x}{(x - 1)^2} = \frac{x(x - 2)}{(x - 1)^2}$$

- Stationary points: $f'(x) = 0 \Rightarrow x = 0, x = 2$
- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
$f'(x)$	+	-	-	+
$f(x)$	↗	↘	↘	↗

The function is increasing on $(-\infty, 0]$, $[2, \infty)$ and decreasing on $[0, 1)$, $(1, 2]$

- At $x = 0$: $f''(0) = -2 < 0$ – a local maximum.

- At $x = 2$: $f''(2) = 2 > 0$ – a local minimum.

- Local maximum $[0; 0]$, local minimum $[2; 4]$

- The second derivative: $f''(x) = \frac{2}{(x-1)^3}$

- The equation $f''(x) = 0$ has no solution.

- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(-\infty, 1)$	$(1, \infty)$
$f''(x)$	$-$	$+$
$f(x)$	$\cap$	$\cup$

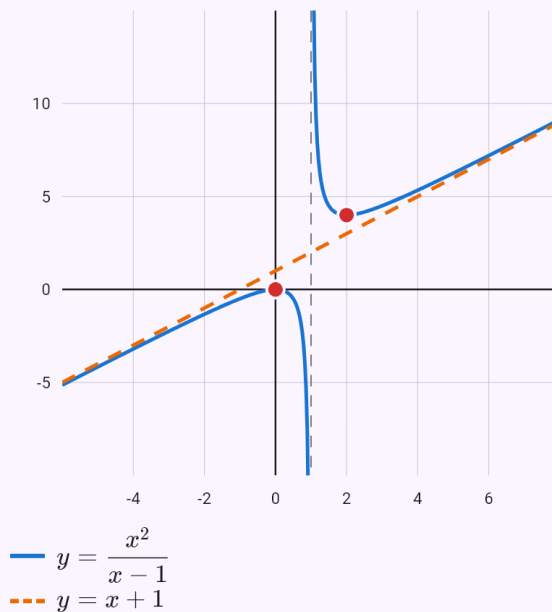
The function is convex on $(1, \infty)$ and concave on $(-\infty, 1)$

- The function has no points of inflection.

- Range – from the monotonicity, extrema and limits:

$$R(f) = (-\infty, 0] \cup [4, \infty)$$

Graph



Notable points:

• maximum $(0, 0)$ • minimum $(2, 4)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

A periodic function – everything on one period, the results with the period.

Input: $\sin 2x + 2\cos x$ [Analyze function](#)

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = \left[-\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}\right]$

Parity: neither even nor odd

Period: $p = 2\pi$

x-intercepts: $\left[\frac{\pi}{2} + k\pi; 0\right]$

Increasing: $\left[-\frac{7\pi}{6} + 2k\pi, \frac{\pi}{6} + 2k\pi\right]$

Decreasing: $\left[\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi\right]$

Local maximum: $\left[\frac{\pi}{6} + 2k\pi; \frac{3\sqrt{3}}{2}\right]$

Local minimum: $\left[\frac{5\pi}{6} + 2k\pi; -\frac{3\sqrt{3}}{2}\right]$

Convex:

$\left[\frac{\pi}{2} + 2k\pi, \pi + \arcsin\left(\frac{1}{4}\right) + 2k\pi\right], \left[\frac{3\pi}{2} + 2k\pi,$

Concave:

$\left[\pi + \arcsin\left(\frac{1}{4}\right) + 2k\pi, \frac{3\pi}{2} + 2k\pi\right], \left[-\arcsin\left(\frac{1}{4}\right) + 2k\pi,$

Points of inflection:

$\left[\frac{\pi}{2} + k\pi; 0\right], \left[\pi + \arcsin\left(\frac{1}{4}\right) + 2k\pi; -\frac{3\sqrt{15}}{8}\right],$

Steps

- Domain: $D(f) = \mathbb{R}$

- The function is periodic with the period 2π – we examine it on one period (from 0 to 2π) and repeat the results with the period, $k \in \mathbb{Z}$.

- $f(-x) = 2\cos x - \sin(2x)$ – neither $f(x)$ nor $-f(x)$: the function is neither even nor odd.

- Intersections with the x-axis: $f(x) = 0 \Rightarrow x = \frac{\pi}{2} + k\pi$, i.e. $\left[\frac{\pi}{2} + k\pi; 0\right]$

- Intersection with the y-axis: $f(0) = 2$, i.e. $[0; 2]$

- The sign of the function between its zeros and the points outside the domain:

	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, 2\pi)$
$f(x)$	$+$	$-$	$+$

The function is positive for $x \in \left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right)$ and negative for $x \in \left(\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi\right)$

- A periodic function (that isn't constant) has no limit at $\pm\infty$.

- The first derivative: $f'(x) = 2\cos(2x) - 2\sin x$

- Stationary points

$$f'(x) = 0 \Rightarrow x = \frac{\pi}{6} + 2k\pi, x = \frac{5\pi}{6} + 2k\pi, x = \frac{3\pi}{2}$$

- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(0, \frac{\pi}{6})$	$(\frac{\pi}{6}, \frac{5\pi}{6})$	$(\frac{5\pi}{6}, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, 2\pi)$
$f'(x)$	+	-	+	+
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$

The function is increasing on $[-\frac{7\pi}{6} + 2k\pi, \frac{\pi}{6} + 2k\pi]$
and decreasing on $[\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi]$

- Local maximum $[\frac{\pi}{6} + 2k\pi; \frac{3\sqrt{3}}{2}]$, local minimum $[\frac{5\pi}{6} + 2k\pi; -\frac{3\sqrt{3}}{2}]$
- The second derivative: $f''(x) = -2\cos x - 4\sin(2x)$

- Candidates for points of inflection

$$f''(x) = 0 \Rightarrow x = \frac{\pi}{2} + k\pi, x = \pi + \arcsin\left(\frac{1}{4}\right) + 2k$$

- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi + \arcsin(\frac{1}{4}))$	$(\pi + \arcsin(\frac{1}{4}), \frac{3\pi}{2})$
$f''(x)$	-	+	-
$f(x)$	$\cap$	$\cup$	$\cap$

The function is convex on and concave on

$$\left[\frac{\pi}{2} + 2k\pi, \pi + \arcsin\left(\frac{1}{4}\right) + 2k\pi\right], \left[\frac{3\pi}{2} + 2k\pi, 2\pi - \arcsin\left(\frac{1}{4}\right) + 2k\pi\right], \left[\pi + \arcsin\left(\frac{1}{4}\right) + 2k\pi, \frac{3\pi}{2} + 2k\pi\right], \left[-\arcsin\left(\frac{1}{4}\right) + 2k\pi, \frac{\pi}{2} + 2k\pi\right]$$

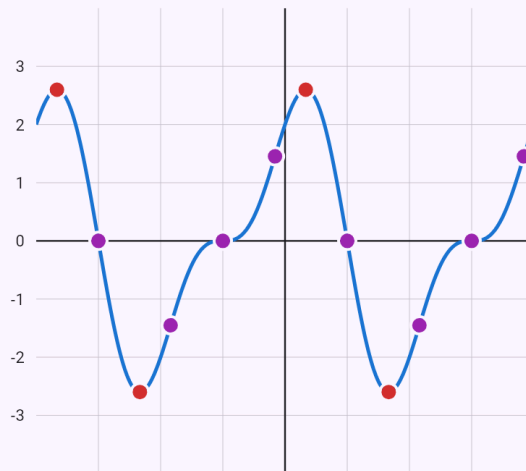
- Points of inflection

$$\left[\frac{\pi}{2} + k\pi; 0\right], \left[\pi + \arcsin\left(\frac{1}{4}\right) + 2k\pi; -\frac{3\sqrt{15}}{8}\right], \left[2\pi - \arcsin\left(\frac{1}{4}\right) + 2k\pi; 0\right]$$

- Range – from the monotonicity, extrema and limits:

$$R(f) = \left[-\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}\right]$$

Graph



$$-3\pi/2 \quad -\pi \quad -\pi/2 \quad 0 \quad \pi/2 \quad \pi \quad 3\pi/2$$

$$y = \sin 2x + 2\cos x$$

Notable points:

- maximum $\left(-\frac{11\pi}{6}, \frac{3\sqrt{3}}{2}\right)$ • inflection $\left(-\frac{3\pi}{2}, 0\right)$
- minimum $\left(-\frac{7\pi}{6}, -\frac{3\sqrt{3}}{2}\right)$
- inflection $\left(-\pi + \arcsin\left(\frac{1}{4}\right), -\frac{3\sqrt{15}}{8}\right)$
- inflection $\left(-\frac{\pi}{2}, 0\right)$
- inflection $\left(-\arcsin\left(\frac{1}{4}\right), \frac{3\sqrt{15}}{8}\right)$
- maximum $\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$ • inflection $\left(\frac{\pi}{2}, 0\right)$
- minimum $\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2}\right)$
- inflection $\left(\pi + \arcsin\left(\frac{1}{4}\right), -\frac{3\sqrt{15}}{8}\right)$
- inflection $\left(\frac{3\pi}{2}, 0\right)$
- inflection $\left(2\pi - \arcsin\left(\frac{1}{4}\right), \frac{3\sqrt{15}}{8}\right)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

Tangent – vertical asymptotes, range $\mathbb{R}$.

Input: tg x

Analyze function

Result

Domain: $D(f) = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\}$

Range: $R(f) = \mathbb{R}$

Parity: odd

Period: $p = \pi$

x-intercepts: $[k\pi; 0]$

Increasing: $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi \right)$

Convex: $\left[k\pi, \frac{\pi}{2} + k\pi \right)$

Concave: $\left(\frac{\pi}{2} + k\pi, \pi + k\pi \right]$

Points of inflection: $[k\pi; 0]$

Asymptotes: $x = \frac{\pi}{2} + k\pi$

Steps

- Domain – conditions $x \neq \frac{\pi}{2} + k\pi$:
 $D(f) = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\}$
- The function is periodic with the period π – we examine it on one period (from 0 to π) and repeat the results with the period, $k \in \mathbb{Z}$.
- $f(-x) = -\tan x = -f(x)$ – the function is odd (its graph is symmetric about the origin).

• Intersections with the x-axis: $f(x) = 0 \Rightarrow x = k\pi$, i.e. $[k\pi; 0]$

• Intersection with the y-axis: $f(0) = 0$, i.e. $[0; 0]$

• The sign of the function between its zeros and the points outside the domain:

	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi)$
$f(x)$	+	-

The function is positive for $x \in \left(k\pi, \frac{\pi}{2} + k\pi \right)$ and negative for $x \in \left(\frac{\pi}{2} + k\pi, \pi + k\pi \right)$

• A periodic function (that isn't constant) has no limit at $\pm\infty$.

• The limits at the ends of the domain

$$\lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \infty$$

$$\lim_{x \rightarrow \frac{\pi}{2}^+} f(x) = -\infty$$

• Vertical asymptotes (an infinite limit there): $x = \frac{\pi}{2} + k\pi$

• The first derivative: $f'(x) = \frac{1}{\cos^2 x}$

• The equation $f'(x) = 0$ has no solution – there are no stationary points.

• The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi)$
$f'(x)$	+	+
$f(x)$	$\nearrow$	$\nearrow$

The function is increasing on $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi \right)$

• The function has no local extrema.

• The second derivative: $f''(x) = \frac{2 \tan x}{\cos^2 x}$

• Candidates for points of inflection:

$$f''(x) = 0 \Rightarrow x = k\pi$$

• The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi)$
$f''(x)$	+	-
$f(x)$	$\cup$	$\cap$

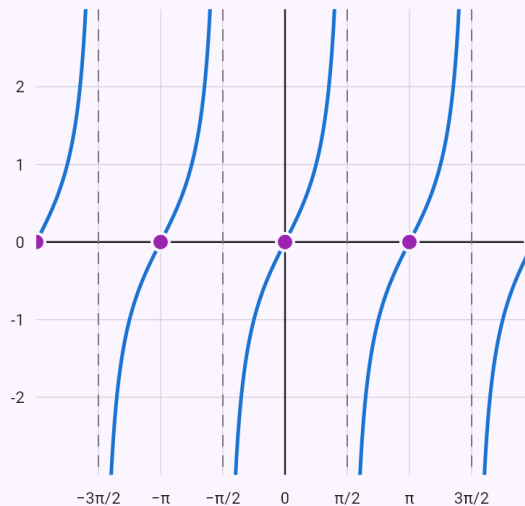
The function is convex on $\left[k\pi, \frac{\pi}{2} + k\pi \right)$ and concave on $\left(\frac{\pi}{2} + k\pi, \pi + k\pi \right]$

• Points of inflection: $[k\pi; 0]$

• Range – from the monotonicity, extrema and limits:

$$R(f) = \mathbb{R}$$

Graph



$y = \text{tg } x$

Notable points:

- inflection $(-2\pi, 0)$
- inflection $(-\pi, 0)$
- inflection $(0, 0)$
- inflection $(\pi, 0)$
- inflection $(2\pi, 0)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

Hyperbolic tangent – horizontal asymptotes $y = \pm 1$.

Input: $\tanh x$ Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = (-1, 1)$

Parity: odd

x-intercepts: $[0; 0]$

Increasing: $\mathbb{R}$

Convex: $(-\infty, 0]$

Concave: $[0, \infty)$

Points of inflection: $[0; 0]$

Asymptotes: $y = -1$, $y = 1$

Steps

- Domain: $D(f) = \mathbb{R}$
- $f(-x) = \tanh(-x) = -\tanh x = -f(x)$ – the function is odd (its graph is symmetric about the origin).
- Intersections with the x-axis: $f(x) = 0 \Rightarrow x = 0$, i.e. $[0; 0]$
- Intersection with the y-axis: $f(0) = 0$, i.e. $[0; 0]$
- The sign of the function between its zeros and the points outside the domain:

	$(-\infty, 0)$	$(0, \infty)$
$f(x)$	-	+

The function is positive for $x \in (0, \infty)$ and negative for $x \in (-\infty, 0)$

- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(-\infty, 0)$	$(0, \infty)$
$f''(x)$	+	-
$f(x)$	∪	∩

The function is convex on $(-\infty, 0]$ and concave on $[0, \infty)$

- At $x = 0$: $f'''(0) = -2 < 0$ – the first non-zero derivative (from the third on) is of an odd order: a point of inflection.
- Points of inflection: $[0; 0]$
- Range – from the monotonicity, extrema and limits:
 $R(f) = (-1, 1)$

- The limits at the ends of the domain

$$\lim_{x \rightarrow -\infty} f(x) = -1$$
$$\lim_{x \rightarrow \infty} f(x) = 1$$

- The function has no vertical asymptotes.

- Asymptote at $-\infty$: $k = \lim_{x \rightarrow -\infty} \frac{f(x)}{x} = 0$,
 $q = \lim_{x \rightarrow -\infty} f(x) = -1$ – the line $y = -1$

- Asymptote at $+\infty$: $k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0$,
 $q = \lim_{x \rightarrow \infty} f(x) = 1$ – the line $y = 1$

- The first derivative: $f'(x) = \frac{1}{\cosh^2 x}$

- The equation $f'(x) = 0$ has no solution – there are no stationary points.

- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(-\infty, \infty)$
$f'(x)$	+
$f(x)$	↗

The function is increasing on $\mathbb{R}$

- The function has no local extrema.

- The second derivative: $f''(x) = -\frac{2 \tanh x}{\cosh^2 x}$

- Candidates for points of inflection: $f''(x) = 0 \Rightarrow x = 0$

Only the derivative is periodic – the extrema values grow with k.

Input: $x + 2\sin x$

Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Parity: odd

Period: $p = 2\pi$

x-intercepts: $[0; 0]$

Increasing: $\left[-\frac{2\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2k\pi\right]$

Decreasing: $\left[\frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi\right]$

Local maximum: $\left[\frac{2\pi}{3} + 2k\pi; \sqrt{3} + \frac{2\pi}{3} + 2k\pi\right]$

Local minimum: $\left[\frac{4\pi}{3} + 2k\pi; \frac{4\pi}{3} - \sqrt{3} + 2k\pi\right]$

Convex: $[\pi + 2k\pi, 2\pi + 2k\pi]$

Concave: $[2k\pi, \pi + 2k\pi]$

Points of inflection:

$[2k\pi; 2k\pi], [\pi + 2k\pi; \pi + 2k\pi]$

Steps

- Domain: $D(f) = \mathbb{R}$
- The function itself isn't periodic, but its derivative is (period 2π) – the monotonicity, extrema and convexity repeat with this period, $k \in \mathbb{Z}$.
- $f(-x) = -x - 2\sin x = -f(x)$ – the function is odd (its graph is symmetric about the origin).

- Intersections with the x-axis: $f(x) = 0 \Rightarrow x = 0$, i.e. $[0; 0]$

- Intersection with the y-axis: $f(0) = 0$, i.e. $[0; 0]$

- The limits at the ends of the domain

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

- The first derivative: $f'(x) = 2\cos x + 1$

- Stationary points

$$f'(x) = 0 \Rightarrow x = \frac{2\pi}{3} + 2k\pi, x = \frac{4\pi}{3} + 2k\pi$$

- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(0, \frac{2\pi}{3})$	$(\frac{2\pi}{3}, \frac{4\pi}{3})$	$(\frac{4\pi}{3}, 2\pi)$
$f'(x)$	+	-	+
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

The function is increasing on $\left[-\frac{2\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2k\pi\right]$

and decreasing on $\left[\frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi\right]$

- Local maximum $\left[\frac{2\pi}{3} + 2k\pi; \sqrt{3} + \frac{2\pi}{3} + 2k\pi\right]$, local minimum $\left[\frac{4\pi}{3} + 2k\pi; \frac{4\pi}{3} - \sqrt{3} + 2k\pi\right]$

- The second derivative: $f''(x) = -2\sin x$

- Candidates for points of inflection:

$$f''(x) = 0 \Rightarrow x = k\pi$$

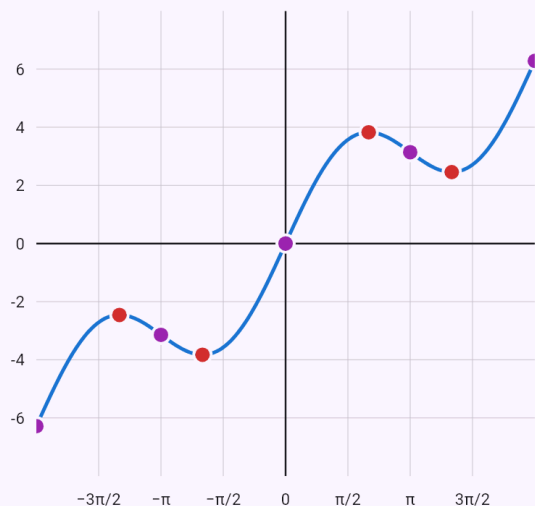
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(0, \pi)$	$(\pi, 2\pi)$
$f''(x)$	$-$	$+$
$f(x)$	$\cap$	$\cup$

The function is convex on $[\pi + 2k\pi, 2\pi + 2k\pi]$ and concave on $[2k\pi, \pi + 2k\pi]$

- Points of inflection: $[2k\pi; 2k\pi]$, $[\pi + 2k\pi; \pi + 2k\pi]$

Graph



$$y = x + 2 \sin x$$

Notable points:

- inflection $(-2\pi, -2\pi)$
- maximum $\left(-\frac{4\pi}{3}, \sqrt{3} - \frac{4\pi}{3}\right)$
- minimum $\left(-\frac{2\pi}{3}, -\sqrt{3} - \frac{2\pi}{3}\right)$
- maximum $\left(\frac{2\pi}{3}, \sqrt{3} + \frac{2\pi}{3}\right)$
- minimum $\left(\frac{4\pi}{3}, \frac{4\pi}{3} - \sqrt{3}\right)$
- inflection $(-\pi, -\pi)$
- inflection $(0, 0)$
- inflection (π, π)
- inflection $(2\pi, 2\pi)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

Absolute value – corners, one-sided derivatives.

Input: $|x^2 - 1|$

Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = [0, \infty)$

Parity: even

x-intercepts: $[-1; 0], [1; 0]$

Increasing: $[-1, 0], [1, \infty)$

Decreasing: $(-\infty, -1], [0, 1]$

Local maximum: $[0; 1]$

Local minimum: $[-1; 0], [1; 0]$

Convex: $(-\infty, -1], [1, \infty)$

Concave: $[-1, 1]$

Steps

• Domain: $D(f) = \mathbb{R}$

• Remove the absolute values by the signs of their arguments on each interval

$$f(x) = \begin{cases} x^2 - 1, & x \in (-\infty, -1) \\ 1 - x^2, & x \in [-1, 1] \\ x^2 - 1, & x \in [1, \infty) \end{cases}$$

• – the function is even (its graph is symmetric about the y-axis).

$$f(-x) = |(-x)^2 - 1| = |x^2 - 1| = f(x)$$

• Intersections with the x-axis:

$$f(x) = 0 \Rightarrow x = -1, x = 1, \text{ i.e. } [-1; 0], [1; 0]$$

• Intersection with the y-axis: $f(0) = 1$, i.e. $[0; 1]$

• The sign of the function between its zeros and the points outside the domain:

	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$f(x)$	+	+	+

The function is positive for

$x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ and negative for $x \in \emptyset$

• The limits at the ends of the domain

$$\lim_{x \rightarrow -\infty} f(x) = \infty$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

• The function has no vertical asymptotes.

• No asymptote at $-\infty$: $k = \lim_{x \rightarrow -\infty} \frac{f(x)}{x} = -\infty$

• No asymptote at $+\infty$: $k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \infty$

• The first derivative

$$f'(x) = \frac{2x^3 - 2x}{|x^2 - 1|} = \frac{2x(x-1)(x+1)}{|x^2 - 1|}$$

• Stationary points: $f'(x) = 0 \Rightarrow x = 0$

• At $x = -1$ there is no derivative – the one-sided derivatives differ ($f'_-(-1) = -2$, $f'_+(-1) = 2$), the graph has a corner there.

- At $x = 1$ there is no derivative – the one-sided derivatives differ ($f'_-(1) = -2$, $f'_+(1) = 2$), the graph has a corner there.

- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$f'(x)$	-	+	-	+
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

The function is increasing on $[-1, 0]$, $[1, \infty)$ and decreasing on $(-\infty, -1]$, $[0, 1]$

- At $x = 0$: $f''(0) = -2 < 0$ – a local maximum.
- Local maximum $[0; 1]$, local minimum $[-1; 0]$, $[1; \infty)$
- The second derivative

$$f''(x) = \frac{2(x^2 - 1)^3}{|x^2 - 1|^3} = \frac{2(x - 1)^3(x + 1)^3}{|x^2 - 1|^3}$$

- The equation $f''(x) = 0$ has no solution.
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

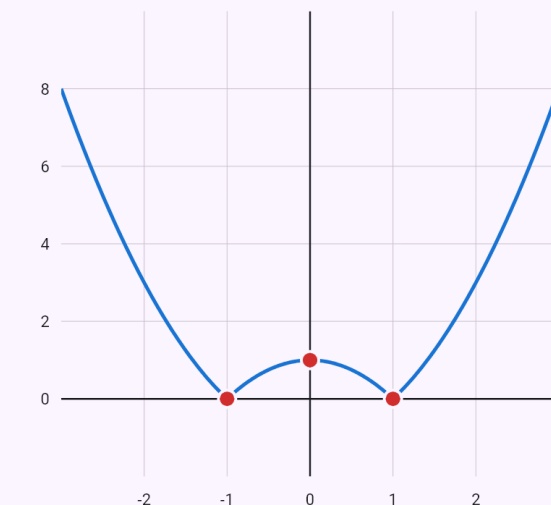
	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$f''(x)$	+	-	+
$f(x)$	$\cup$	$\cap$	$\cup$

The function is convex on $(-\infty, -1]$, $[1, \infty)$ and concave on $[-1, 1]$

- The function has no points of inflection.

- Range – from the monotonicity, extrema and limits:
 $R(f) = [0, \infty)$

Graph



$$y = |x^2 - 1|$$

Notable points:

- minimum $(-1, 0)$
- maximum $(0, 1)$
- minimum $(1, 0)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

Cube root – a vertical tangent at zero.

Input: $x^{1/3}$

Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = \mathbb{R}$

Parity: odd

x-intercepts: $[0; 0]$

Increasing: $\mathbb{R}$

Convex: $(-\infty, 0]$

Concave: $[0, \infty)$

Points of inflection: $[0; 0]$

Steps

- Domain: $D(f) = \mathbb{R}$
- $f(-x) = \sqrt[3]{-x} = -\sqrt[3]{x} = -f(x)$ – the function is odd (its graph is symmetric about the origin).
- Intersections with the x-axis: $f(x) = 0 \Rightarrow x = 0$, i.e. $[0; 0]$
- Intersection with the y-axis: $f(0) = 0$, i.e. $[0; 0]$
- The sign of the function between its zeros and the points outside the domain:

	$(-\infty, 0)$	$(0, \infty)$
$f(x)$	-	+

The function is positive for $x \in (0, \infty)$ and negative for $x \in (-\infty, 0)$

- The limits at the ends of the domain

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

- The function has no vertical asymptotes.
- No asymptote at $-\infty$: $k = \lim_{x \rightarrow -\infty} \frac{f(x)}{x} = 0$, but $q = \lim_{x \rightarrow -\infty} f(x) = -\infty$
- No asymptote at $+\infty$: $k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0$, but $q = \lim_{x \rightarrow \infty} f(x) = \infty$
- The first derivative: $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$
- The equation $f'(x) = 0$ has no solution – there are no stationary points.
- At $x = 0$ the derivative is infinite ($f'_-(0) = \infty$, $f'_+(0) = \infty$) – the graph has a vertical tangent there.
- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	+	+
$f(x)$	$\nearrow$	$\nearrow$

The function is increasing on $\mathbb{R}$

- The function has no local extrema.
- The second derivative: $f''(x) = -\frac{2}{9x\sqrt[3]{x^2}}$

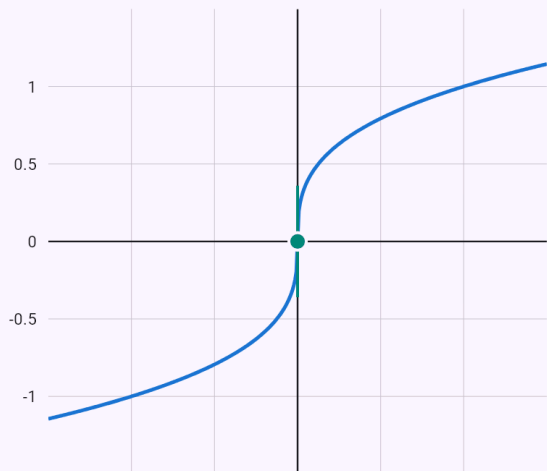
- The equation $f''(x) = 0$ has no solution.
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(-\infty, 0)$	$(0, \infty)$
$f''(x)$	+	-
$f(x)$	∪	∩

The function is convex on $(-\infty, 0]$ and concave on $[0, \infty)$

- Points of inflection: $[0; 0]$
- Range – from the monotonicity, extrema and limits:
 $R(f) = \mathbb{R}$

Graph



-1 -0.5 0 0.5 1

$y = x^{\frac{1}{3}}$

Notable points:

- inflection $(0, 0)$ • vertical tangent $(0, 0)$

Local extrema in red, points of inflection in purple, a hollow circle
– a point where the function isn't defined; asymptotes dashed.

A nested absolute value – by pieces, an even function.

Input: $|-x^2 + 2|x| + 3|$

Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = [0, \infty)$

Parity: even

x-intercepts: $[-3; 0], [3; 0]$

Increasing: $[-3, -1], [0, 1], [3, \infty)$

Decreasing: $(-\infty, -3], [-1, 0], [1, 3]$

Local maximum: $[-1; 4], [1; 4]$

Local minimum: $[-3; 0], [0; 3], [3; 0]$

Convex: $(-\infty, -3], [3, \infty)$

Concave: $[-3, 0], [0, 3]$

Steps

• Domain: $D(f) = \mathbb{R}$

• Remove the absolute values by the signs of their arguments on each interval

$$f(x) = \begin{cases} x^2 + 2x - 3, & x \in (-\infty, -3) \\ -x^2 - 2x + 3, & x \in [-3, 0) \\ -x^2 + 2x + 3, & x \in [0, 3) \\ x^2 - 2x - 3, & x \in [3, \infty) \end{cases}$$

• – the function is even (its graph is symmetric about the y-axis).

$$f(-x) = | -(-x)^2 + 2|(-x)| + 3 | = |-x^2 + 2|x| + 3| :$$

• Intersections with the x-axis:

$$f(x) = 0 \Rightarrow x = -3, x = 3, \text{ i.e. } [-3; 0], [3; 0]$$

• Intersection with the y-axis: $f(0) = 3$, i.e. $[0; 3]$

• The sign of the function between its zeros and the points outside the domain:

$$\begin{array}{c|ccc} f(x) & (-\infty, -3) & (-3, 3) & (3, \infty) \\ \hline & + & + & + \end{array}$$

The function is positive for

$x \in (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ and negative for $x \in \emptyset$

• The limits at the ends of the domain

$$\lim_{x \rightarrow -\infty} f(x) = \infty$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

• The function has no vertical asymptotes.

• No asymptote at $-\infty$: $k = \lim_{x \rightarrow -\infty} \frac{f(x)}{x} = -\infty$

• No asymptote at $+\infty$: $k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \infty$

• The first derivative

$$f'(x) = \frac{-6x^3 + 2x^3|x| + 6x - 2x|x|}{|x| |-x^2 + 2|x| + 3|} = \frac{2x(x-1)(x+1)}{|x| |-x^2 + 2|x| + 3|}$$

• Stationary points: $f'(x) = 0 \Rightarrow x = -1, x = 1$

• At $x = -3$ there is no derivative – the one-sided derivatives differ ($f'_-(-3) = -4, f'_+(-3) = 4$), the graph has a corner there.

- At $x = 0$ there is no derivative – the one-sided derivatives differ ($f'_-(0) = -2$, $f'_+(0) = 2$), the graph has a corner there.
- At $x = 3$ there is no derivative – the one-sided derivatives differ ($f'_-(3) = -4$, $f'_+(3) = 4$), the graph has a corner there.

- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(-\infty, -3)$	$(-3, -1)$	$(-1, 0)$	$(0, 1)$	$(1, 3)$
$f'(x)$	$-$	$+$	$-$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$

The function is increasing on $[-3, -1]$, $[0, 1]$, $[3, \infty)$ and decreasing on $(-\infty, -3]$, $[-1, 0]$, $[1, 3]$

- At $x = -1$: $f''(-1) = -2 < 0$ – a local maximum.
- At $x = 1$: $f''(1) = -2 < 0$ – a local maximum.
- Local maximum $[-1; 4]$, $[1; 4]$, local minimum $[-3; 0]$, $[0; 3]$, $[3; 0]$
- The second derivative

$$f''(x) = \frac{12x^8 + 24x^6 + 36x^6|x| + 2x^6|x|^3 - 84x^4 - 96}{\dots}$$

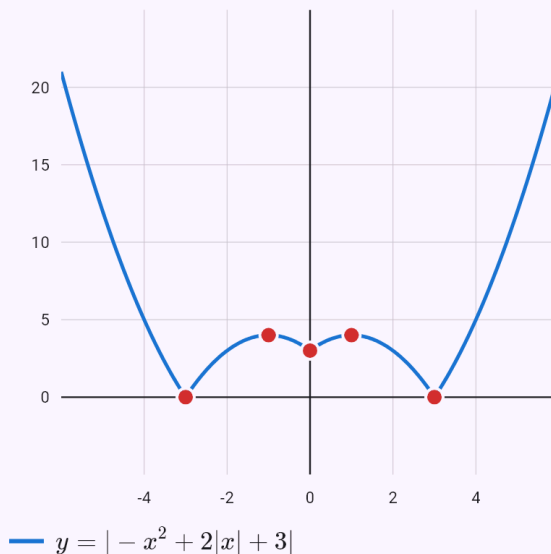
- The equation $f''(x) = 0$ has no solution.
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(-\infty, -3)$	$(-3, 0)$	$(0, 3)$	$(3, \infty)$
$f''(x)$	$+$	$-$	$-$	$+$
$f(x)$	$\cup$	$\cap$	$\cap$	$\cup$

The function is convex on $(-\infty, -3]$, $[3, \infty)$ and concave on $[-3, 0]$, $[0, 3]$

- The function has no points of inflection.
- Range – from the monotonicity, extrema and limits: $R(f) = [0, \infty)$

Graph



Notable points:

- minimum $(-3, 0)$ • maximum $(-1, 4)$
- minimum $(0, 3)$ • maximum $(1, 4)$
- minimum $(3, 0)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

A sum of absolute values – a constant piece and two oblique asymptotes.

Input: $|x - 2| + |x + 3|$

Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = [5, \infty)$

Parity: neither even nor odd

Increasing: $[2, \infty)$

Decreasing: $(-\infty, -3]$

Constant: $[-3, 2]$

Local minimum: $f(x) = 5, x \in [-3, 2]$

Asymptotes: $y = -2x - 1, y = 2x + 1$

Steps

• Domain: $D(f) = \mathbb{R}$

• Remove the absolute values by the signs of their arguments on each interval

$$f(x) = \begin{cases} -2x - 1, & x \in (-\infty, -3) \\ 5, & x \in [-3, 2] \\ 2x + 1, & x \in [2, \infty) \end{cases}$$

• – neither $f(x)$ nor $-f(x)$: the function is neither even nor odd.

$$f(-x) = |(-x) - 2| + |(-x) + 3| = |-x - 2| + |3 - x|$$

• The equation $f(x) = 0$ has no solution – the graph doesn't cross the x-axis.

• Intersection with the y-axis: $f(0) = 5$, i.e. $[0; 5]$

• The sign of the function between its zeros and the points

derivatives differ ($f'_-(2) = 0, f'_+(2) = 2$), the graph has a corner there.

• The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(-\infty, -3)$	$(-3, 2)$	$(2, \infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	$\rightarrow$	$\nearrow$

The function is increasing on $[2, \infty)$ and decreasing on $(-\infty, -3]$. On $[-3, 2]$ it is constant ($f' = 0$).

• Local minimum $f(x) = 5, x \in [-3, 2]$, no local maximum.

• The second derivative: $f''(x) = 0$

• The second derivative is zero – the graph is (part of) a line, neither convex nor concave.

• Range – from the monotonicity, extrema and limits:
 $R(f) = [5, \infty)$

outside the domain:

$$\frac{f(x)}{x} \Big| \begin{matrix} (-\infty, \infty) \\ + \end{matrix}$$

The function is positive for $x \in \mathbb{R}$ and negative for $x \in \emptyset$

• The limits at the ends of the domain

$$\lim_{x \rightarrow -\infty} f(x) = \infty$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

• The function has no vertical asymptotes.

• Asymptote at $-\infty$: $k = \lim_{x \rightarrow -\infty} \frac{f(x)}{x} = -2$,
 $q = \lim_{x \rightarrow -\infty} (f(x) + 2x) = -1$ – the line $y = -2x - 1$

• Asymptote at $+\infty$: $k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = 2$,
 $q = \lim_{x \rightarrow \infty} (f(x) - 2x) = 1$ – the line $y = 2x + 1$

• The first derivative

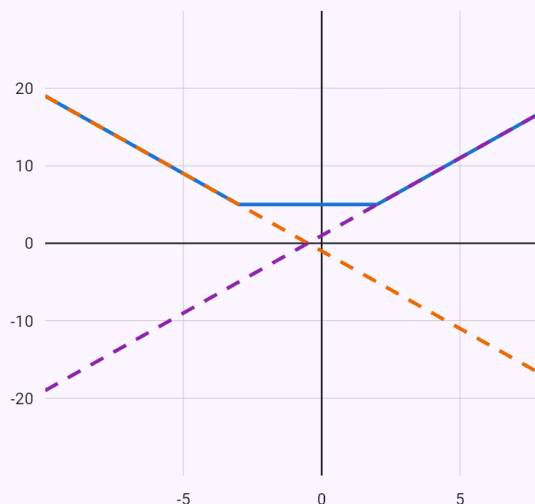
$$f'(x) = \frac{x}{|x-2|} + \frac{x}{|x+3|} - \frac{2}{|x-2|} + \frac{3}{|x+3|} = \frac{x|x-2|}{|x-2|^2} + \frac{x|x+3|}{|x+3|^2} - \frac{2}{|x-2|} + \frac{3}{|x+3|}$$

• The equation $f'(x) = 0$ has no solution – there are no stationary points.

• At $x = -3$ there is no derivative – the one-sided derivatives differ ($f'_-(-3) = -2, f'_+(-3) = 0$), the graph has a corner there.

• At $x = 2$ there is no derivative – the one-sided

Graph



$$\text{— } y = |x - 2| + |x + 3|$$

$$\text{--- } y = -2x - 1$$

$$\text{--- } y = 2x + 1$$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

Known limitations

- A function without even a periodic derivative ($x \cdot \sin x$) is analyzed only numerically on $\langle -2\pi; 2\pi \rangle$, without the period; inverse trigonometric functions ($\arcsin$, $\arctan$) are not supported yet.

Taylor polynomial

The app builds the Taylor (at 0 the Maclaurin) polynomial of a given degree: a table of the derivatives at the point, substitution into the formula and the remainder in Lagrange form. It can also find the degree itself – the smallest one for which the error at a given point or on an interval stays below a given bound. The graph shows the function and the polynomial; a slider changes the degree so you can see the approximation improve.

How to enter it

<code>taylor(sin x, 0, 5)</code>	The function, the point and the degree.
<code>maclaurin(e^x, 4)</code>	The Maclaurin polynomial – the point is 0.
<code>taylor(e^x, 0, error=0.001, x=0.5)</code>	The degree by the error at the point x (on an interval enter r=... instead of x).
<code>ln(1 + x)</code>	A function with the “Taylor polynomial” chip – fill in the point, the degree or the accuracy there.

Examples

A polynomial of degree 4 with the remainder.

Input: `taylor(ln(1 + x), 0, 4)`

Result

$$T_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

Steps

- The Maclaurin polynomial (the Taylor polynomial at 0):

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k, \quad n = 4$$

- The derivatives and their values at a

$$f(x) = \ln(x+1) \quad f(0) = 0$$

$$f'(x) = \frac{1}{x+1} \quad f'(0) = 1$$

$$f''(x) = -\frac{1}{(x+1)^2} \quad f''(0) = -1$$

$$f'''(x) = \frac{2}{(x+1)^3} \quad f'''(0) = 2$$

$$f^{(4)}(x) = -\frac{6}{(x+1)^4} \quad f^{(4)}(0) = -6$$

- Filling in the values of the derivatives

$$\begin{aligned} T_4(x) &= \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4 \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \end{aligned}$$

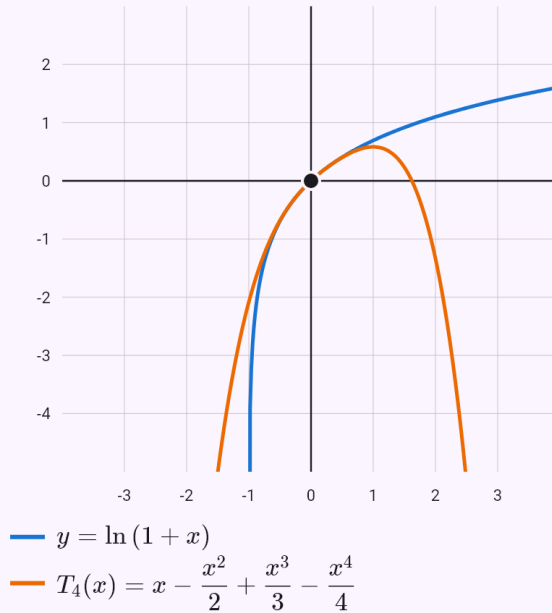
- (0, 0)

Use the slider above the graph to pick the degree – with a higher degree the polynomial approximates the function better, and further from the point.

- The remainder (the error) in Lagrange's form, ξ between a and x: $R_4(x) = \frac{f^{(5)}(\xi)}{5!}x^5 = \frac{24}{(\xi+1)^5 \cdot 5!}x^5$

Graph

Degree: 4



The point around which the polynomial approximates the function:

The degree by the required accuracy – an approximate value of $e^{0.5}$.

Input: `taylor(e^x, 0, error=0.001, x=0.5)`

Result

$$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$f(0.5) \approx 1.6484$$

Steps

- The Maclaurin polynomial (the Taylor polynomial at 0):

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k, \quad n = 4$$

- The derivatives and their values at a

$$f(x) = e^x \quad f(0) = 1$$

$$f'(x) = e^x \quad f'(0) = 1$$

$$f''(x) = e^x \quad f''(0) = 1$$

$$f'''(x) = e^x \quad f'''(0) = 1$$

$$f^{(4)}(x) = e^x \quad f^{(4)}(0) = 1$$

- Filling in the values of the derivatives

$$\begin{aligned} T_4(x) &= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \end{aligned}$$

- The remainder (the error) in Lagrange's form, ξ between

$$a \text{ and } x: R_4(x) = \frac{f^{(5)}(\xi)}{5!} x^5 = \frac{e^\xi}{5!} x^5$$

- The lowest degree n whose remainder estimate at x

keeps within the error

$$M_k = \max_{t \in [0, 0.5]} |f^{(k)}(t)|, \quad h = |x - a| = 0.5$$

$$n = 1: |R_1| \leq \frac{1.6487}{2!} \cdot 0.5^2 \approx 0.2063 > 10^{-3}$$

$$n = 2: |R_2| \leq \frac{1.6487}{3!} \cdot 0.5^3 \approx 0.0344 > 10^{-3}$$

$$n = 3: |R_3| \leq \frac{1.6487}{4!} \cdot 0.5^4 \approx 4.3 \cdot 10^{-3} > 10^{-3}$$

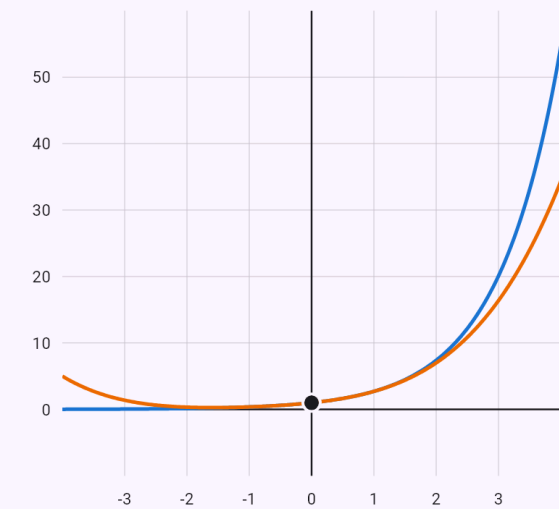
$$n = 4: |R_4| \leq \frac{1.6487}{5!} \cdot 0.5^5 \approx 4.3 \cdot 10^{-4} \leq 10^{-3}$$

- The approximate value:

$$f(0.5) \approx T_4(0.5) = \frac{211}{128} \approx 1.6484$$

Graph

Degree: 4



— $y = e^x$

— $T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

The point around which the polynomial approximates the function:

• $(0, 1)$

Use the slider above the graph to pick the degree – with a higher degree the polynomial approximates the function better, and further from the point.

Indefinite integrals

The app integrates with the methods taught at school and says at every step what it does. It checks the result by differentiating (the check is the last step) and draws the integrand and the antiderivative.

Roots of powers of the variable ($\sqrt[3]{x^4}$, $\sqrt{x^3}$) are rewritten as powers with rational exponents; an absolute value ($|\cos x|$, $\sqrt{1 - \sin^2 x}$) is written with the sign function sgn .

If an integral has no elementary antiderivative (such as $e^{(-x^2)}$), the app says so and gives it in terms of a special function (erf , Si , ...).

How to enter it

<code>int(x e^x, x)</code>	An integral with the variable.
<code>int(x^2 sin x)</code>	Without the variable when there is only one.
<code>∫ x^2 dx</code>	With the integral sign, also from a photo.
<code>x e^x</code>	A function with the “Integrate” chip.

Basic integrals and substitution

The integral of a sum term by term, constants taken out, table integrals (also with a linear inner function) and substitution when the integrand contains a function together with the derivative of its inside.

Examples

Term by term from the table.

Input: `int(3x^2 - 2/x + e^x)`

Result

$$\int \left(3x^2 + e^x - \frac{2}{x} \right) dx = x^3 + e^x - 2$$

Steps

- The integral of a sum is the sum of the integrals

$$\int \left(3x^2 + e^x - \frac{2}{x} \right) dx = \int 3x^2 dx + \int e^x dx + \int -\frac{2}{x} dx$$

- The constant taken out of the integral:

$$\int 3x^2 dx = 3 \int x^2 dx$$

- From the table of basic integrals: $\int x^2 dx = \frac{x^3}{3}$

- From the table of basic integrals: $\int e^x dx = e^x$

- The constant taken out of the integral:

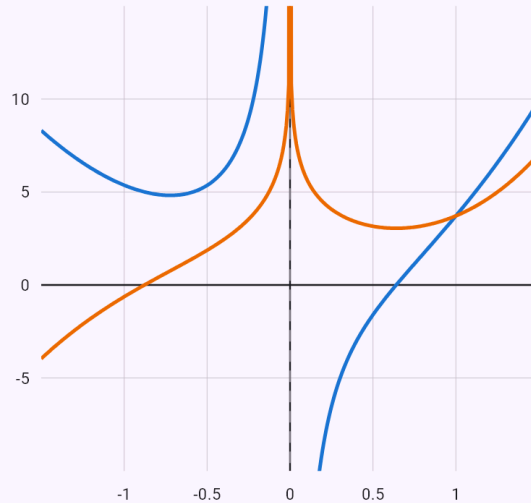
$$\int -\frac{2}{x} dx = -2 \int \frac{1}{x} dx$$

- From the table of basic integrals: $\int \frac{1}{x} dx = \ln |x|$

- Check – the result differentiated gives the integrand:

$$(x^3 + e^x - 2 \ln |x|)' = 3x^2 + e^x - \frac{2}{x}$$

Graph



$$\begin{aligned} \text{— } f(x) &= 3x^2 + e^x - \frac{2}{x} \\ \text{— } F(x) &= x^3 + e^x - 2 \ln |x| \end{aligned}$$

The integrand f and the antiderivative F for $C = 0$ (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

The substitution $t = x^2$.

Input: `int(2x cos(x^2))`

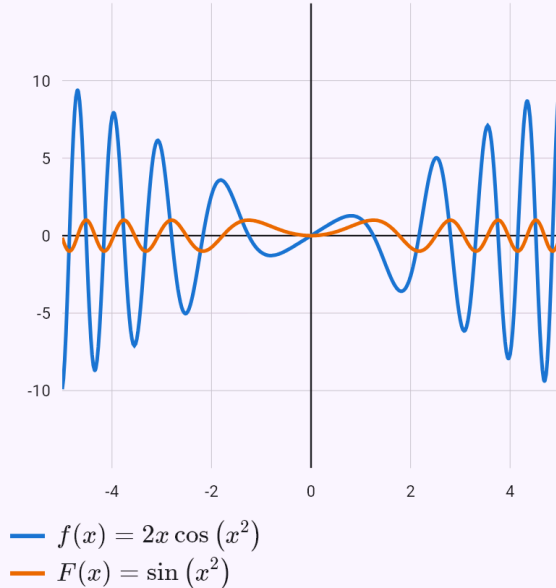
Result

$$\int 2x \cos(x^2) dx = \sin(x^2) + C$$

Steps

- The constant taken out of the integral:
$$\int 2x \cos(x^2) dx = 2 \int x \cos(x^2) dx$$
- Substituting $t = x^2, dt = 2x dx$:
$$\int x \cos(x^2) dx = \int \frac{\cos t}{2} dt$$
- The constant taken out of the integral:
$$\int \frac{\cos t}{2} dt = \frac{1}{2} \int \cos t dt$$
- From the table of basic integrals: $\int \cos t dt = \sin t$
- Back to the original variable: $\frac{\sin t}{2} = \frac{\sin(x^2)}{2}$
- Check – the result differentiated gives the integrand:
$$(\sin(x^2))' = 2x \cos(x^2)$$

Graph



The integrand f and the antiderivative F for $C = 0$ (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

An integral with constants a, b .

Input: `int(1/(a - bx)^3)`

Result

$$\int \frac{1}{(a - bx)^3} dx = \frac{1}{2b(bx - a)^2} + C$$

Conditions: $b \neq 0$

Steps

- From the table – the inner function is linear ($ax + b$), so the result is divided by a :
$$\int \frac{1}{(a - bx)^3} dx = \frac{1}{2b(bx - a)^2}$$
- Check – the result differentiated gives the integrand:
$$\left(\frac{1}{2b(bx - a)^2} \right)' = \frac{1}{(a - bx)^3}$$

Integration by parts

The app integrates a product of a polynomial with an exponential, a trigonometric function or a logarithm by parts, repeatedly if needed; integrals like $e^x \sin x$ by cyclic integration by parts.

Examples

Integration by parts twice in a row.

Input: `int(x^2 e^x)`

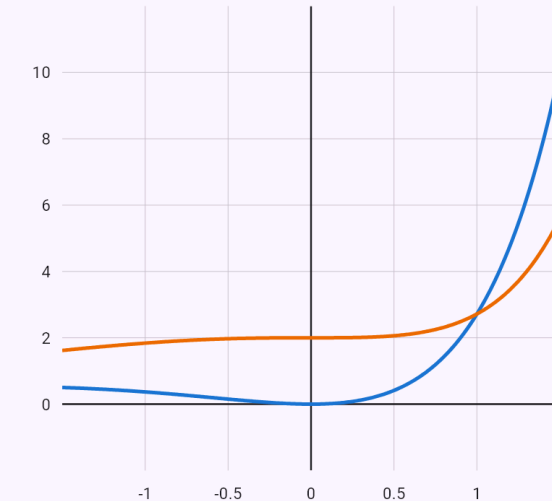
Result

$$\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + C$$

Steps

- By parts ($\int u \cdot v' = u \cdot v - \int u' \cdot v$), $u = x^2$, $v' = e^x$, so $u' = 2x$, $v = e^x$:
$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$$
- The constant taken out of the integral:
$$\int 2x e^x dx = 2 \int x e^x dx$$
- By parts ($\int u \cdot v' = u \cdot v - \int u' \cdot v$), $u = x$, $v' = e^x$, so $u' = 1$, $v = e^x$:
$$\int x e^x dx = x e^x - \int e^x dx$$
- From the table of basic integrals: $\int e^x dx = e^x$
- Check – the result differentiated gives the integrand:
$$(x^2 e^x - 2x e^x + 2e^x)' = x^2 e^x$$

Graph



— $f(x) = x^2 e^x$
— $F(x) = x^2 e^x - 2x e^x + 2e^x$

The integrand f and the antiderivative F for $C = 0$ (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

Rational functions

The app divides a rational function if needed, factors the denominator and splits the function into partial fractions, which it integrates into logarithms and arctangents.

A denominator that can't be factored over the rationals ($x^4 + 1$, $x^5 + 1$) is factored over the reals – each pair of complex conjugate roots gives a quadratic factor with no real roots.

Examples

Partial fractions.

Input: `int((x + 1)/(x^2 - 3x + 2))`

Result

$$\int \frac{x+1}{x^2-3x+2} dx = 3 \ln|x-2| - 2 \ln|x-1|$$

Steps

- The rational function split into a polynomial and partial fractions:

$$\int \frac{x+1}{x^2-3x+2} dx = \int \left(\frac{3}{x-2} - \frac{2}{x-1} \right) dx$$

- The integral of a sum is the sum of the integrals

$$\int \left(\frac{3}{x-2} - \frac{2}{x-1} \right) dx = \int \frac{3}{x-2} dx + \int -\frac{2}{x-1} dx$$

- The constant taken out of the integral:

$$\int \frac{3}{x-2} dx = 3 \int \frac{1}{x-2} dx$$

- From the table of basic integrals:

$$\int \frac{1}{x-2} dx = \ln|x-2|$$

- The constant taken out of the integral:

$$\int -\frac{2}{x-1} dx = -2 \int \frac{1}{x-1} dx$$

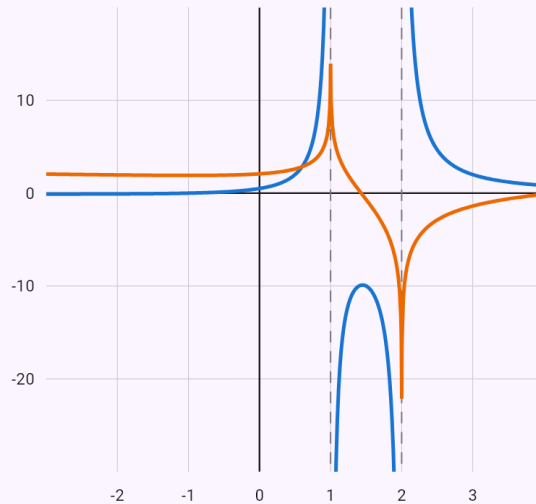
- From the table of basic integrals:

$$\int \frac{1}{x-1} dx = \ln|x-1|$$

- Check – the result differentiated gives the integrand

$$(3 \ln|x-2| - 2 \ln|x-1|)' = \frac{3}{x-2} - \frac{2}{x-1} = \frac{x+1}{x^2-3x+2}$$

Graph



$$f(x) = \frac{x+1}{x^2-3x+2}$$

$$F(x) = 3 \ln|x-2| - 2 \ln|x-1|$$

The integrand f and the antiderivative F for $C = 0$ (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

Factoring the denominator $x^4 + 1$ over the reals.

Input: `int(1/(x^4 + 1))`

Result

$$\int \frac{1}{x^4 + 1} dx = \frac{\sqrt{2}}{4} \arctan \left(\sqrt{2} \left(x - \frac{1}{x} \right) \right) + \frac{\sqrt{2}}{4} \arctan \left(\sqrt{2} (x + 1) \right) + \frac{\sqrt{2}}{4} \arctan \left(\sqrt{2} (x - 1) \right) + \frac{\sqrt{2}}{4} \arctan \left(\sqrt{2} \left(x - \frac{1}{x} \right) \right)$$

Steps

- Factor the denominator over $\mathbb{R}$: its complex roots come in conjugate pairs $r, \bar{r}$, and each pair gives a quadratic factor $(x - r)(x - \bar{r}) = x^2 - 2\operatorname{Re} r \cdot x + |r|^2$ with no real roots: $x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$

- The rational function split into a polynomial and partial fractions

$$\int \frac{1}{x^4 + 1} dx = \int \left(\frac{\sqrt{2}x + 2}{4(x^2 + \sqrt{2}x + 1)} + \frac{2 - \sqrt{2}x}{4(x^2 - \sqrt{2}x + 1)} \right) dx$$

- The integral of a sum is the sum of the integrals

$$\int \left(\frac{\sqrt{2}x + 2}{4(x^2 + \sqrt{2}x + 1)} + \frac{2 - \sqrt{2}x}{4(x^2 - \sqrt{2}x + 1)} \right) dx = \int \frac{\sqrt{2}x + 2}{4(x^2 + \sqrt{2}x + 1)} dx + \int \frac{2 - \sqrt{2}x}{4(x^2 - \sqrt{2}x + 1)} dx$$

- The denominator has no real roots: a multiple of its derivative split off the numerator gives $\ln$, the rest – after completing the square – gives $\arctan$

$$\int \frac{\sqrt{2}x + 2}{4(x^2 + \sqrt{2}x + 1)} dx = \frac{\sqrt{2}}{8} \int \frac{2x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} dx + \frac{\sqrt{2}}{8} \int \frac{2}{x^2 + \sqrt{2}x + 1} dx$$

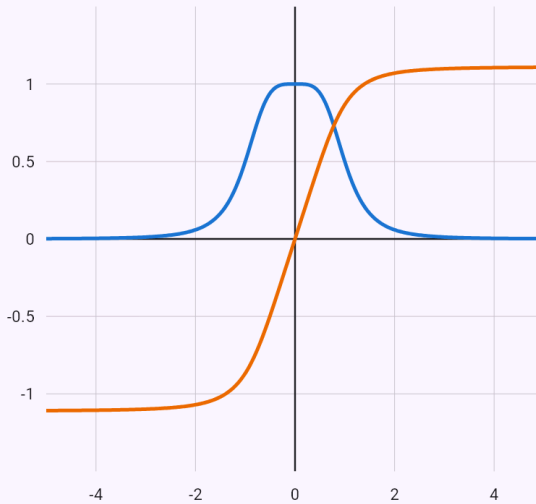
- The denominator has no real roots: a multiple of its derivative split off the numerator gives $\ln$, the rest – after completing the square – gives $\arctan$

$$\int \frac{2 - \sqrt{2}x}{4(x^2 - \sqrt{2}x + 1)} dx = -\frac{\sqrt{2}}{8} \int \frac{2x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} dx + \frac{\sqrt{2}}{8} \int \frac{2}{x^2 - \sqrt{2}x + 1} dx$$

- Check – the result differentiated gives the integrand

$$\left(\frac{\sqrt{2}}{4} \arctan \left(\sqrt{2} \left(x - \frac{1}{x} \right) \right) + \frac{\sqrt{2}}{4} \arctan \left(\sqrt{2} (x + 1) \right) + \frac{\sqrt{2}}{4} \arctan \left(\sqrt{2} (x - 1) \right) + \frac{\sqrt{2}}{4} \arctan \left(\sqrt{2} \left(x - \frac{1}{x} \right) \right) \right)'$$

Graph



$$f(x) = \frac{1}{x^4 + 1}$$

$$F(x) = \frac{\sqrt{2}}{4} \arctan \left(\sqrt{2} \left(x - \frac{1}{x} \right) \right) + \frac{\sqrt{2}}{4} \arctan \left(\sqrt{2} (x + 1) \right) + \frac{\sqrt{2}}{4} \arctan \left(\sqrt{2} (x - 1) \right) + \frac{\sqrt{2}}{4} \arctan \left(\sqrt{2} \left(x - \frac{1}{x} \right) \right)$$

The integrand f and the antiderivative F for $C = 0$ (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

Trigonometric functions

Identities first ($\sin^2 x = (1 - \cos 2x)/2$, products into sums), for an odd power the substitution $t = \sin x$ or $t = \cos x$, for a rational function of sine and cosine a substitution by parity, and the Weierstrass substitution $t = \tan(x/2)$ only as the last resort.

Examples

An odd power of sine – the substitution $t = \cos x$.

Input: `int(sin(x)^3 cos(x)^2)`

Result

$$\int \cos^2 x \sin^3 x dx = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} .$$

Steps

- An odd power – the substitution $t = \cos x$,

$$\sin^2 x = 1 - \cos^2 x$$

$$dt = -\sin x dx$$

$$\int \cos^2 x \sin^3 x dx = \int (t^4 - t^2) dt$$

- The integral of a sum is the sum of the integrals:

$$\int (t^4 - t^2) dt = \int t^4 dt + \int -t^2 dt$$

- From the table of basic integrals: $\int t^4 dt = \frac{t^5}{5}$

- The constant taken out of the integral:

$$\int -t^2 dt = -\int t^2 dt$$

- From the table of basic integrals: $\int t^2 dt = \frac{t^3}{3}$

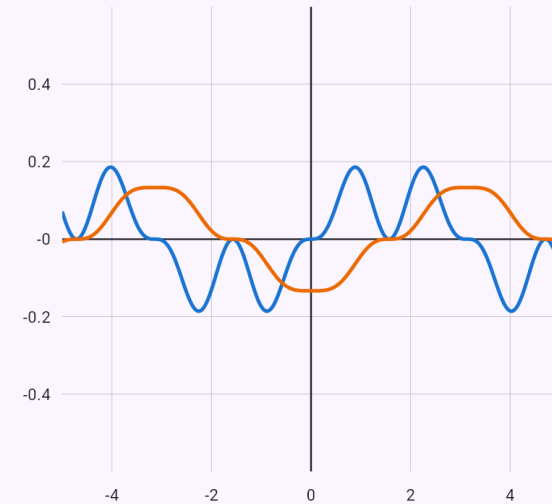
- Back to the original variable:

$$t = \cos x : \quad \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3}$$

- Check – the result differentiated gives the integrand:

$$\left(\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} \right)' = \cos^2 x \sin^3 x$$

Graph



$$\begin{aligned} \text{--- } f(x) &= \cos^2 x \sin^3 x \\ \text{--- } F(x) &= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} \end{aligned}$$

The integrand f and the antiderivative F for $C = 0$ (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

Hyperbolic functions

Table integrals ($\int \sinh x dx = \cosh x + C$, $\int 1/\cosh^2 x dx = \tanh x + C$, ...), $\int \tanh x dx$ by the logarithm of the derivative, by parts ($x \sinh x$, $\operatorname{arsinh} x$), squares by the formulas $\sinh^2 x = (\cosh 2x - 1)/2$, $\tanh^2 x = 1 - 1/\cosh^2 x$; the rest by the definition through exponentials.

Examples

The formula $\sinh^2 x = (\cosh 2x - 1)/2$.

Input: `int(sinh(x)^2)`

Result

$$\int \sinh^2 x \, dx = -\frac{x}{2} + \frac{\sinh(2x)}{4} + C$$

Steps

- Using the formula $\sinh^2 \alpha = \frac{\cosh 2\alpha - 1}{2}$:

$$\int \sinh^2 x \, dx = \int \frac{\cosh(2x) - 1}{2} \, dx$$

- The integral of a sum is the sum of the integrals

$$\int \left(\frac{\cosh(2x)}{2} - \frac{1}{2} \right) dx = \int -\frac{1}{2} \, dx + \int \frac{\cosh(2x)}{2} \, dx$$

- From the table of basic integrals: $\int -\frac{1}{2} \, dx = -\frac{x}{2}$

- The constant taken out of the integral:

$$\int \frac{\cosh(2x)}{2} \, dx = \frac{1}{2} \int \cosh(2x) \, dx$$

- From the table – the inner function is linear ($ax + b$), so the result is divided by a : $\int \cosh(2x) \, dx = \frac{\sinh(2x)}{2}$

- Check – the result differentiated gives the integrand

$$\left(-\frac{x}{2} + \frac{\sinh(2x)}{4} \right)' = \frac{\cosh(2x) - 1}{2} = \sinh^2 x$$

Roots

Integrals with a root of a linear fractional function by substitution, with a root of a quadratic by an Euler substitution, and binomial integrals by Chebyshev's conditions – when none of them holds, the app says the integral is not elementary.

Examples

An Euler substitution.

Input: `int(sqrt(x^2 + 1))`

Result

$$\int \sqrt{x^2 + 1} dx = \frac{x\sqrt{x^2 + 1}}{2} + \frac{\ln|x + \sqrt{x^2 + 1}|}{2}$$

Steps

- Euler's substitution (type 1) $\sqrt{x^2 + 1} = t - x$ - the integral becomes one of a rational function of t

$$x = \frac{t^2 - 1}{2t}, \quad dx = \frac{t^2 + 1}{2t^2} dt, \quad \sqrt{x^2 + 1} = \frac{t^2 + 1}{2t}$$

$$\int \sqrt{x^2 + 1} dx = \int \frac{(t^2 + 1)^2}{4t^3} dt$$

- The constant taken out of the integral

$$\int \frac{(t^2 + 1)^2}{4t^3} dt = \frac{1}{4} \int \frac{(t^2 + 1)^2}{t^3} dt$$

- The rational function split into a polynomial and partial fractions

$$\int \frac{(t^2 + 1)^2}{t^3} dt = \int \left(t + \frac{2}{t} + \frac{1}{t^3} \right) dt$$

- The integral of a sum is the sum of the integrals

$$\int \left(t + \frac{2}{t} + \frac{1}{t^3} \right) dt = \int t dt + \int \frac{2}{t} dt + \int \frac{1}{t^3} dt$$

- From the table of basic integrals: $\int t dt = \frac{t^2}{2}$

- The constant taken out of the integral:

$$\int \frac{2}{t} dt = 2 \int \frac{1}{t} dt$$

- From the table of basic integrals: $\int \frac{1}{t} dt = \ln|t|$

- From the table of basic integrals: $\int \frac{1}{t^3} dt = -\frac{1}{2t^2}$

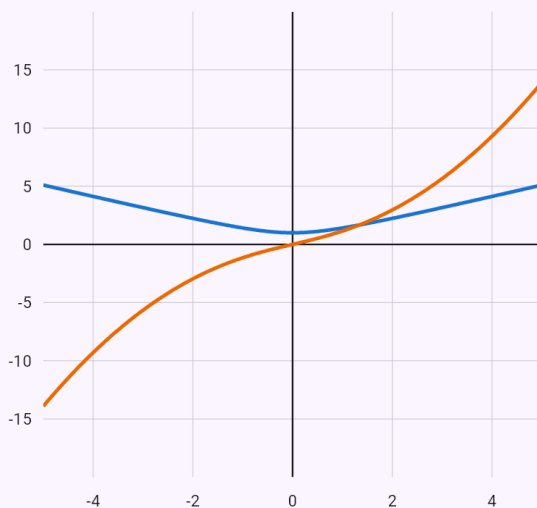
- Back to the original variable

$$t = x + \sqrt{x^2 + 1} : \quad \frac{\ln|x + \sqrt{x^2 + 1}|}{2} + \frac{(x^2 + 1)}{2}$$

- Check - the result differentiated gives the integrand

$$\left(\frac{x\sqrt{x^2 + 1}}{2} + \frac{\ln|x + \sqrt{x^2 + 1}|}{2} \right)' = \sqrt{x^2 + 1}$$

Graph



$$\text{Blue line: } f(x) = \sqrt{x^2 + 1}$$

$$\text{Orange line: } F(x) = \frac{x\sqrt{x^2 + 1}}{2} + \frac{\ln|x + \sqrt{x^2 + 1}|}{2}$$

The integrand f and the antiderivative F for $C = 0$ (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

Known limitations

- For some integrals we find the result without the working – the app says so in the steps.
- A computation without steps that takes too long is stopped – the app says so.

Definite integrals and their applications

The app computes a definite integral by the Newton–Leibniz formula (the fundamental theorem of calculus) and draws the area under the graph – the part above the x-axis in a different colour from the part below. It also handles improper integrals and applications: area, arc length, volume and surface area of a solid of revolution, also for curves given parametrically and in polar coordinates.

Definite integral

The app finds the antiderivative as for an indefinite integral and substitutes the limits. With a substitution it converts the limits right away. If the integral has no elementary antiderivative, it computes it numerically.

How to enter it

<code>int(x^2, x, 0, 3)</code>	An integral with the variable and the limits.
<code>\int_{0}^{\pi} \sin x\,dx</code>	From a photo or in LaTeX.
<code>x^2</code>	A function with the “Integrate” chip – fill in the limits in the fields.

Examples

A substitution with the limits converted.

Input: `int(3x^2 cos(x^3), x, 0, 1)`

Result

$$\int_0^1 3x^2 \cos(x^3) dx = \sin 1 \approx 0.8415$$

Steps

- The constant taken out of the integral:

$$\int 3x^2 \cos(x^3) dx = 3 \int x^2 \cos(x^3) dx$$

- Substituting $t = x^3$, $dt = 3x^2 dx$:

$$\int x^2 \cos(x^3) dx = \int \frac{\cos t}{3} dt$$

- The constant taken out of the integral:

$$\int \frac{\cos t}{3} dt = \frac{1}{3} \int \cos t dt$$

- From the table of basic integrals: $\int \cos t dt = \sin t$

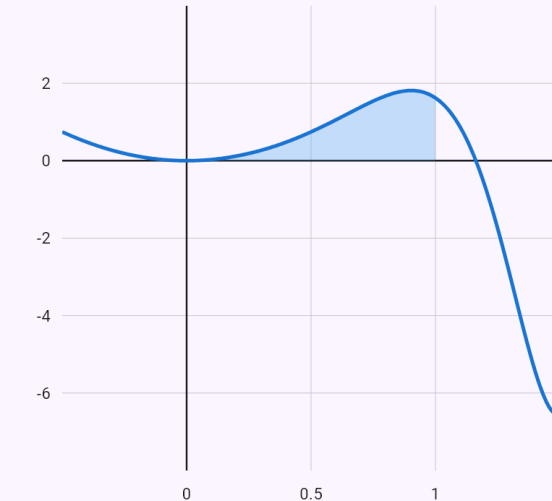
- With the substitution, the limits are recomputed too – no need to go back to the original variable:

$$x = 0 \Rightarrow t = 0, \quad x = 1 \Rightarrow t = 1$$

- The Newton–Leibniz formula, $F(b) - F(a)$

$$\int_0^1 3x^2 \cos(x^3) dx = [\sin t]_0^1 = \sin 1 - 0 = \sin 1$$

Graph



— $f(x) = 3x^2 \cos(x^3)$

■ above the x-axis (+) ■ below the x-axis (-)

The definite integral is the area between the graph and the x-axis – the part above the axis counts positively, the part below negatively.

Improper integrals

The app handles an infinite limit or an unbounded function with a limit of the antiderivative and says whether the integral converges or diverges. If that limit cannot be computed, it decides by a comparison test.

Examples

An infinite upper limit.

Input: `int(1/x^2, x, 1, oo)`

Result

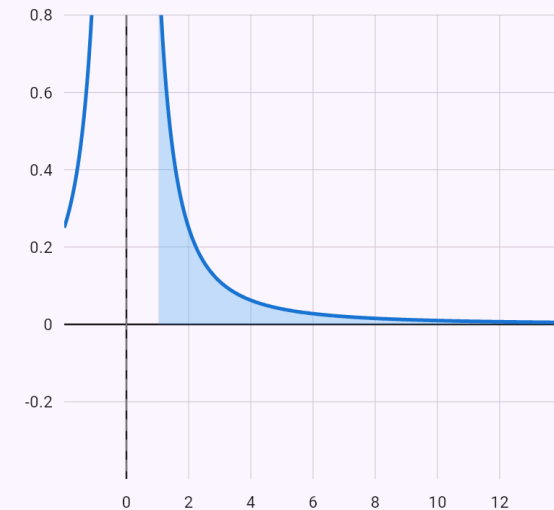
$$\int_1^{\infty} \frac{1}{x^2} dx = 1$$

Steps

- From the table of basic integrals: $\int \frac{1}{x^2} dx = -\frac{1}{x}$
- An improper integral – at an infinite limit and where the function is unbounded we take the limit of the antiderivative. The limits are finite, the integral converges

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{x \rightarrow \infty} \left(-\frac{1}{x} \right) - (-1) = 0 - (-1) = 1$$

Graph



$$f(x) = \frac{1}{x^2}$$

■ above the x-axis (+) ■ below the x-axis (-)

The definite integral is the area between the graph and the x-axis
– the part above the axis counts positively, the part below negatively.

A function unbounded at the lower limit.

Input: `int(1/sqrt(x), x, 0, 1)`

Result

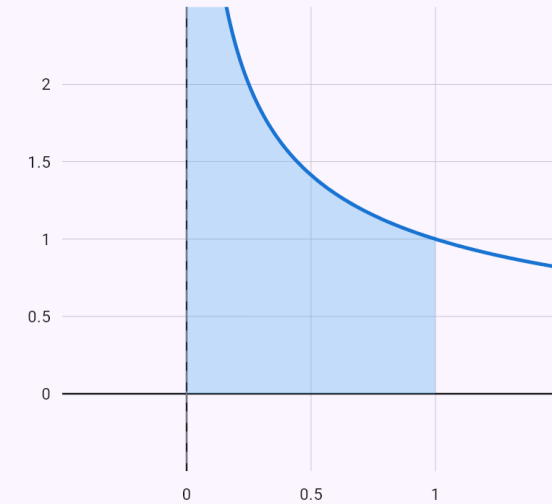
$$\int_0^1 \frac{1}{\sqrt{x}} dx = 2$$

Steps

- From the table of basic integrals: $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x}$
- An improper integral – at an infinite limit and where the function is unbounded we take the limit of the antiderivative. The limits are finite, the integral converges

$$\int_0^1 \frac{1}{\sqrt{x}} dx = 2 - \lim_{x \rightarrow 0^+} 2\sqrt{x} = 2 - 0 = 2$$

Graph



$$f(x) = \frac{1}{\sqrt{x}}$$

■ above the x-axis (+) ■ below the x-axis (-)

The definite integral is the area between the graph and the x-axis – the part above the axis counts positively, the part below negatively.

Area under a curve and between curves

The app computes the area between a graph and the x-axis, or between two curves, as the integral of the absolute value of the difference – parts above and below both count as positive. You don't have to enter the limits: it finds them as the intersections of the curves.

How to enter it

`area(x^2, x)`

The area between two curves, limits from the intersections.

`area(sin x, 0, pi)`

The area between a curve and the x-axis on an interval.

`x^2`

A function with the “Integrate” chip → “Area between curves”; fill in the second curve and the limits in the fields.

Examples

The area between a parabola and a line.

Input: x^2 Area between curves, second curve x

Result

$$S = \frac{1}{6}$$

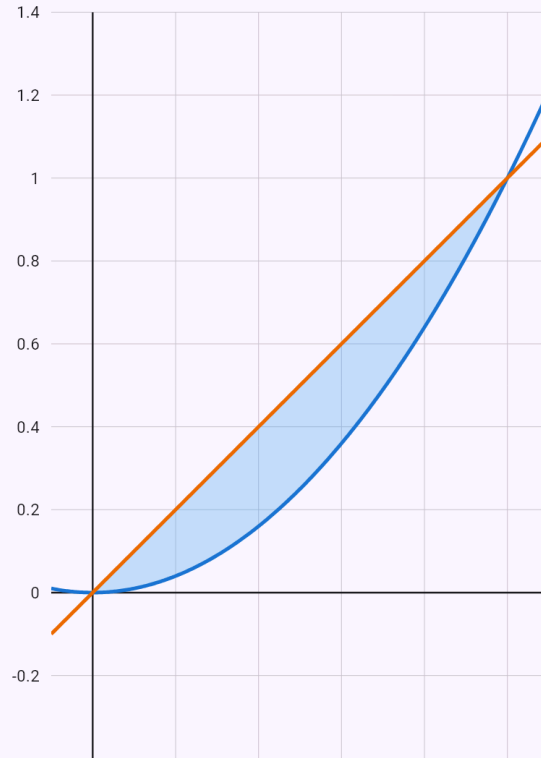
Steps

- The limits are the curves' intersections:
 $x^2 = x \Rightarrow x = 0, x = 1$
- The area between the curves (parts above and below count positively), $S = \int_a^b |f(x) - g(x)| dx$:
$$S = \int_0^1 |x^2 - x| dx$$
- The expression in absolute value doesn't change its sign on the interval – the absolute value removed by it:
$$\int_0^1 |x^2 - x| dx = \int_0^1 (-x^2 + x) dx$$
- The integral of a sum is the sum of the integrals:
$$\int (-x^2 + x) dx = \int -x^2 dx + \int x dx$$
- The constant taken out of the integral:
$$\int -x^2 dx = - \int x^2 dx$$
- From the table of basic integrals: $\int x^2 dx = \frac{x^3}{3}$
- From the table of basic integrals: $\int x dx = \frac{x^2}{2}$

- The Newton–Leibniz formula, $F(b) - F(a)$

$$\int_0^1 (-x^2 + x) dx = \left[-\frac{x^3}{3} + \frac{x^2}{2} \right]_0^1 = \frac{1}{6} - 0 = \frac{1}{6}$$

Graph



0 0.2 0.4 0.6 0.8 1

— $f(x) = x^2$
— $g(x) = x$

The shaded region – its area is the result (parts above and below the axis count positively).

Arc length

The app computes the length of a graph on an interval by the formula $l = \int \sqrt{1 + f'^2} dx$. It tries to simplify the root (for a semicircle it comes out as arcsine); where the integral has no elementary antiderivative, the result is numerical.

How to enter it

`length( $x^{(3/2)}$ , 0, 4)`

The arc length on an interval.

Examples

The length of a semicircle.

Input: `sqrt(1 - x^2)` Arc length, limits -1 and 1

Result

$$l = \pi \approx 3.1416$$

Steps

- The length of the curve

$$l = \int_a^b \sqrt{1 + (f'(x))^2} dx, \quad f'(x) = -\frac{x}{\sqrt{1-x^2}}$$

$$l = \int_{-1}^1 \sqrt{1 + \left(-\frac{x}{\sqrt{1-x^2}}\right)^2} dx = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx$$

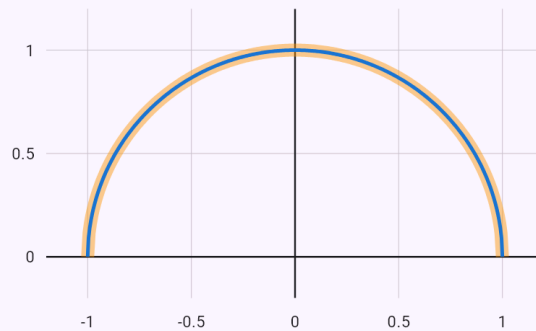
- From the table of basic integrals:

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x$$

- An improper integral – at an infinite limit and where the function is unbounded we take the limit of the antiderivative. The limits are finite, the integral converges

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = \lim_{x \rightarrow 1^-} \arcsin x - \lim_{x \rightarrow -1^+} \arcsin x = \frac{\pi}{2}$$

Graph



$$f(x) = \sqrt{1-x^2}$$

The highlighted arc – its length is the result.

Volume of a solid of revolution

The app computes the volume of a solid of revolution about the x-axis by the formula $V = \pi \int f^2 dx$ (between two curves as washers), about the y-axis by the shell method. It also shows the solid in 3D – rotate it with your finger.

How to enter it

`volume(sqrt(x), 0, 4)`

The volume of the solid rotated about the x-axis.

`volume(x^2, 0, 2, y)`

Rotation about the y-axis.

Examples

A paraboloid of revolution.

Input: `sqrt(x)` Volume of a solid, limits 0 and 4

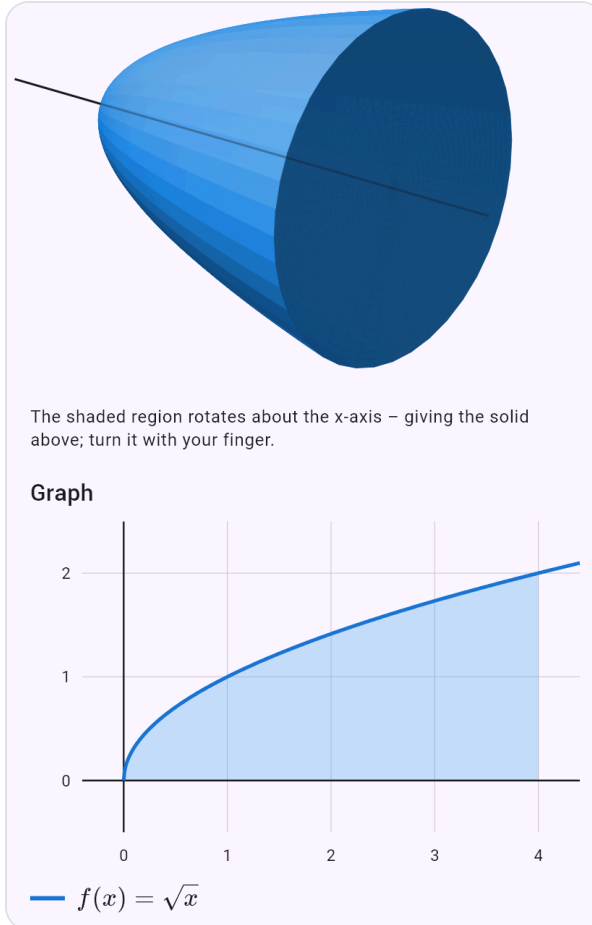
Result

$$V = 8\pi \approx 25.1327$$

Steps

- The volume of the solid of revolution about the x-axis,
$$V = \pi \int_a^b f^2(x) dx: V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x dx$$
- From the table of basic integrals: $\int x dx = \frac{x^2}{2}$
- The Newton–Leibniz formula, $F(b) - F(a)$:
$$\int_0^4 x dx = \left[\frac{x^2}{2} \right]_0^4 = 8 - 0 = 8$$
- The integral's value times the constant in front of it:
$$V = \pi \cdot 8 = 8\pi$$

Solid of revolution



Surface area of a solid of revolution

The app computes the lateral surface area by the formula $S = 2\pi \int f \sqrt{1 + f'^2} dx$; the product under the root often simplifies (for a sphere it comes out constant).

How to enter it

`surface(sqrt(4 - x^2), -2, 2)`

The surface of the solid rotated about the x-axis.

Examples

The surface of a sphere of radius 2.

Input: `sqrt(4 - x^2)` Surface of a solid, limits -2 and 2

Result

$$S = 16\pi \approx 50.2655$$

Steps

- The lateral surface of the solid of revolution about the x-axis

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx, \quad f'(x) = -\frac{x}{\sqrt{4-x^2}}$$

$$S = 2\pi \int_{-2}^2 \sqrt{4-x^2} \sqrt{1 + \left(-\frac{x}{\sqrt{4-x^2}}\right)^2} dx = 2\pi \int_{-2}^2 2 dx$$

- From the table of basic integrals: $\int 2 dx = 2x$

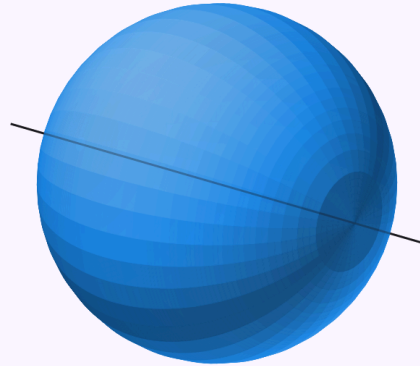
- The Newton-Leibniz formula, $F(b) - F(a)$:

$$\int_{-2}^2 2 dx = [2x]_{-2}^2 = 4 - (-4) = 8$$

- The integral's value times the constant in front of it:

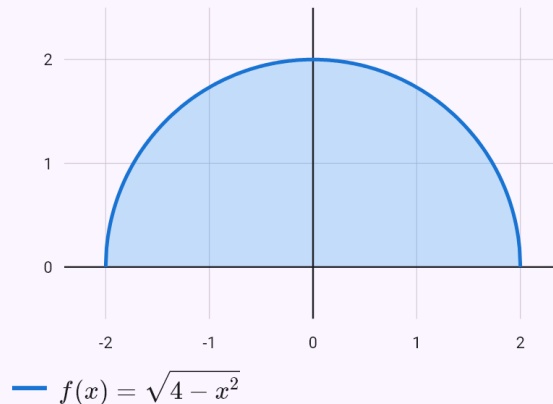
$$S = 2\pi \cdot 8 = 16\pi$$

Solid of revolution



The shaded region rotates about the x-axis – giving the solid above; turn it with your finger.

Graph



Parametric curves and curves in polar coordinates

Write a curve $x = \varphi(t)$, $y = \psi(t)$ on two lines, a curve in polar coordinates as $r = \rho(\varphi)$. The “Integrate” chip then offers the area, length, volume and surface; the limits of the parameter are optional (without them from 0 to 2π). Volume and surface are computed by rotating the part of the curve above the x-axis about the x-axis.

How to enter it

$$x = t - \sin(t)$$

$$y = 1 - \cos(t)$$

A parametric curve on two lines.

$$r = 1 + \cos(\phi)$$

A curve in polar coordinates (also $r^2 = \dots$ for a lemniscate).

length($x=\cos(t)^3$, $y=\sin(t)^3$, t , 0 , 2π)

The written form with the parameter and the limits.

Examples

The area under an arch of a cycloid.

Input: $x = t - \sin(t)$

$y = 1 - \cos(t)$ **Area**

Result

$$S = 3\pi \approx 9.4248$$

Steps

- The derivatives by the parameter:

$$x'(t) = 1 - \cos t, \quad y'(t) = \sin t$$

- The area under a parametric curve (inside a closed one)

$$\text{for } t \in [0, 2\pi], S = \left| \int_{\alpha}^{\beta} y(t) x'(t) dt \right|:$$

$$S = \left| \int_0^{2\pi} (-2 \cos t + \cos^2 t + 1) dt \right|$$

- The integral of a sum is the sum of the integrals

$$\int (-2 \cos t + \cos^2 t + 1) dt = \int 1 dt + \int -2 \cos t dt -$$

- From the table of basic integrals: $\int 1 dt = t$

- The constant taken out of the integral:

$$\int -2 \cos t dt = -2 \int \cos t dt$$

- From the table of basic integrals: $\int \cos t dt = \sin t$

- Using the formula $\cos^2 t = \frac{1 + \cos 2t}{2}$:

$$\int \cos^2 t dt = \int \left(\frac{\cos(2t)}{2} + \frac{1}{2} \right) dt$$

- The integral of a sum is the sum of the integrals

$$\int \left(\frac{\cos(2t)}{2} + \frac{1}{2} \right) dt = \int \frac{1}{2} dt + \int \frac{\cos(2t)}{2} dt$$

- From the table of basic integrals: $\int \frac{1}{2} dt = \frac{t}{2}$

- The constant taken out of the integral:

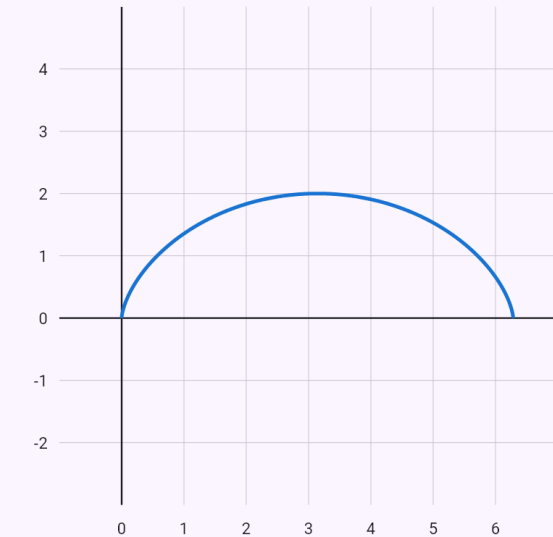
$$\int \frac{\cos(2t)}{2} dt = \frac{1}{2} \int \cos(2t) dt$$

- From the table – the inner function is linear ($ax + b$), so the result is divided by a : $\int \cos(2t) dt = \frac{\sin(2t)}{2}$

- The Newton–Leibniz formula, $F(b) - F(a)$

$$\int_0^{2\pi} (-2 \cos t + \cos^2 t + 1) dt = \left[\frac{3t}{2} - 2 \sin t + \frac{\sin(2t)}{4} \right]$$

Graph



$$x = t - \sin t, \quad y = 1 - \cos t$$

The curve given parametrically (or in polar coordinates), drawn for the chosen limits of the parameter.

The length of a cardioid.

Input: $r = 1 + \cos(\phi)$ Arc length

Result

$$l = 8$$

Steps

- The derivatives by the parameter: $\rho'(\varphi) = -\sin \varphi$

- The length of a polar curve for $\varphi \in [0, 2\pi]$,

$$l = \int_{\alpha}^{\beta} \sqrt{\rho^2(\varphi) + \rho'^2(\varphi)} d\varphi$$

$$l = \int_0^{2\pi} \sqrt{(\cos \varphi + 1)^2 + (-\sin \varphi)^2} d\varphi = \int_0^{2\pi} 2 \left| \cos \left(\frac{\varphi}{2} \right) \right| d\varphi$$

- We split the interval at the zero points of the expressions in absolute value (π) and remove the absolute value in each part by its sign

$$\int_0^{2\pi} 2 \left| \cos \left(\frac{\varphi}{2} \right) \right| d\varphi = \int_0^{\pi} 2 \cos \left(\frac{\varphi}{2} \right) d\varphi + \int_{\pi}^{2\pi} -2 \cos \left(\frac{\varphi}{2} \right) d\varphi$$

- The constant taken out of the integral

$$\int 2 \cos \left(\frac{\varphi}{2} \right) d\varphi = 2 \int \cos \left(\frac{\varphi}{2} \right) d\varphi$$

- From the table – the inner function is linear ($ax + b$), so the result is divided by a : $\int \cos \left(\frac{\varphi}{2} \right) d\varphi = 2 \sin \left(\frac{\varphi}{2} \right)$

- The Newton–Leibniz formula, $F(b) - F(a)$

$$\int_0^{\pi} 2 \cos \left(\frac{\varphi}{2} \right) d\varphi = \left[4 \sin \left(\frac{\varphi}{2} \right) \right]_0^{\pi} = 4 - 0 = 4$$

- The constant taken out of the integral

$$\int -2 \cos \left(\frac{\varphi}{2} \right) d\varphi = -2 \int \cos \left(\frac{\varphi}{2} \right) d\varphi$$

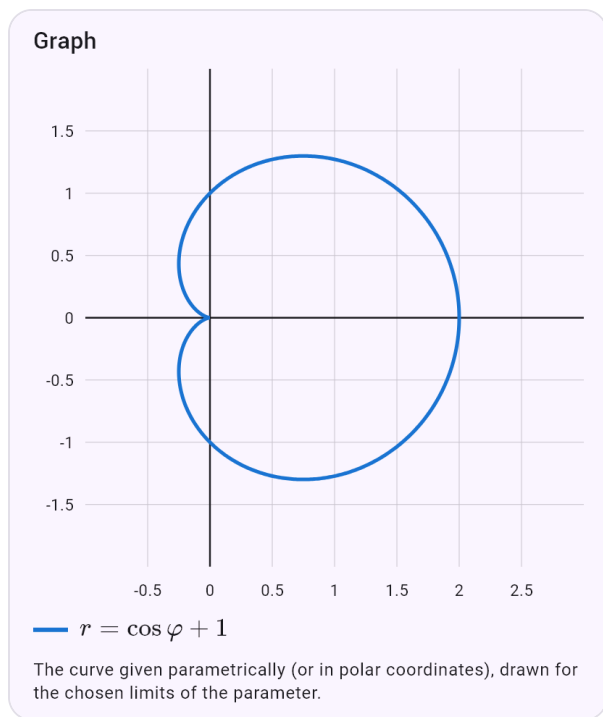
- From the table – the inner function is linear ($ax + b$), so the result is divided by a : $\int \cos \left(\frac{\varphi}{2} \right) d\varphi = 2 \sin \left(\frac{\varphi}{2} \right)$

- The Newton–Leibniz formula, $F(b) - F(a)$

$$\int_{\pi}^{2\pi} -2 \cos \left(\frac{\varphi}{2} \right) d\varphi = \left[-4 \sin \left(\frac{\varphi}{2} \right) \right]_{\pi}^{2\pi} = 0 - (-4) = 4$$

- The parts' values added up:

$$\int_0^{2\pi} 2 \left| \cos \left(\frac{\varphi}{2} \right) \right| d\varphi = 4 + 4 = 8$$



Curves given by an equation

For an equation in x and y ($x^3 + y^3 = 3xy$) the “Integrate” chip offers the area, the length, the volume and the surface. Without limits it takes the whole closed curve: a circle or an ellipse in parametric form by $\cos$ and $\sin$, a loop through the origin (the folium of Descartes) by the lines $y = tx$ – the loop’s area then exactly from the sectors, the length, volume and surface numerically. With limits in x it uses the upper branch $y = f(x)$.

How to enter it

$$x^3 + y^3 = 3xy$$

A closed curve – the “Integrate” chip without limits.

$$y^2 = x^3$$

A curve that isn’t closed – the “Integrate” chip with limits in x (the upper branch $y = x^{3/2}$).

Examples

The area of the folium's loop – by the lines $y = tx$ and the sectors, $S = 3/2$.

Input: $x^3 + y^3 = 3xy$ [Integrate → Area](#)

Result

$$S = \frac{3}{2}$$

Steps

- The loop through the origin in parametric form by the lines $y = tx$ – substituted into the equation, x expressed

$$y = tx \Rightarrow x = \frac{3t}{t^3 + 1}, \quad y = \frac{3t^2}{t^3 + 1}, \quad t \in (0; \infty)$$

- The loop's area from the sectors from the origin: $S = \frac{1}{2} \int (x \cdot y' - y \cdot x') dt$, and on the line $y = tx$, $x \cdot y' - y \cdot x' = x^2$

$$S = \frac{1}{2} \int_0^\infty x^2(t) dt = \int_0^\infty \frac{9t^2}{2(t^3 + 1)^2} dt$$

- The constant taken out of the integral

$$\int \frac{9t^2}{2(t^3 + 1)^2} dt = \frac{9}{2} \int \frac{t^2}{(t^3 + 1)^2} dt$$

- Substituting $u = t^3 + 1$, $du = 3t^2 dt$:

$$\int \frac{t^2}{(t^3 + 1)^2} dt = \int \frac{1}{3u^2} du$$

- The constant taken out of the integral:

$$\int \frac{1}{3u^2} du = \frac{1}{3} \int \frac{1}{u^2} du$$

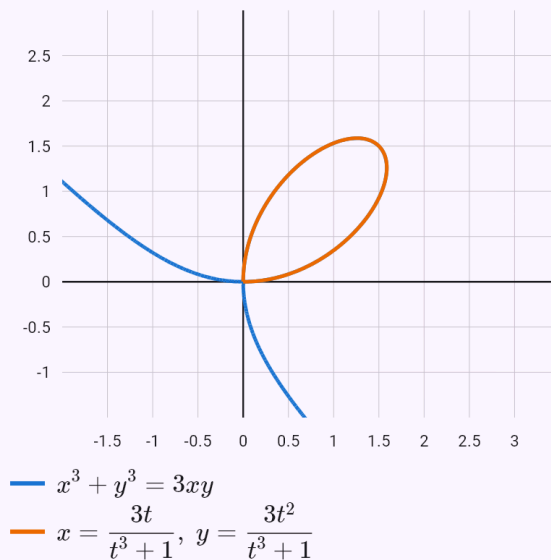
- From the table of basic integrals: $\int \frac{1}{u^2} du = -\frac{1}{u}$

- Back to the original variable: $-\frac{1}{3u} = -\frac{1}{3(t^3 + 1)}$

- An improper integral – at an infinite limit and where the function is unbounded we take the limit of the antiderivative. The limits are finite, the integral converges

$$\int_0^\infty \frac{9t^2}{2(t^3 + 1)^2} dt = \lim_{t \rightarrow \infty} \left(-\frac{3}{2(t^3 + 1)} \right) - \left(-\frac{3}{2} \right) = \frac{3}{2}$$

Graph



The whole curve given by the equation; the part the computation uses is highlighted.

Known limitations

- The arc length of $y = \sqrt[3]{x}$ (and similar curves) is only computed numerically – it leads to an elliptic integral, which has no elementary antiderivative.
- For parametric and polar curves the shaded area and the 3D solid are still missing; rotation only about the x-axis.
- Rotation about the y-axis only for a region on one side of the axis.
- For implicit curves without limits only a circle, an ellipse and a loop through the origin are supported; rotation only about the x-axis.

Infinite series

For a numerical series the app first checks the necessary condition for convergence (the term test) and then picks one test by the form of the series – geometric series, $\sum 1/n^p$, the ratio, root, limit comparison, integral or alternating series (Leibniz) test; for a series with alternating signs it distinguishes absolute and conditional convergence. Where possible it sums the series. The graph shows the terms and the partial sums.

Numerical series

The app finds the sum of geometric and telescoping series, series $\sum P(n)q^n$ and well-known series ($\sum 1/n^2 = \pi^2/6$); otherwise it computes it numerically. A finite sum is added up the school way – telescoping, arithmetic, geometric, otherwise term by term. An equation with the sum of a geometric series is solved too, with the condition of convergence $|q| < 1$ checked.

How to enter it

<code>sum(1/n^2)</code>	The general term; the index n from 1.
<code>sum(1/n^2, n, 1, oo)</code>	With the index and the starting value.
<code>\sum_{n=1}^{\infty} \frac{1}{n^2}</code>	From a photo or in LaTeX.
<code>1/n^2</code>	The general term with the “Series” chip – fill in the starting index in the field.
<code>sum(2k+1, k, 1, 100)</code>	A finite sum – with the index, the lower and the upper bound.

Examples

The series $\sum 1/n^2$ – converges, the sum is $\pi^2/6$.

Input: `sum(1/n^2)`

Result

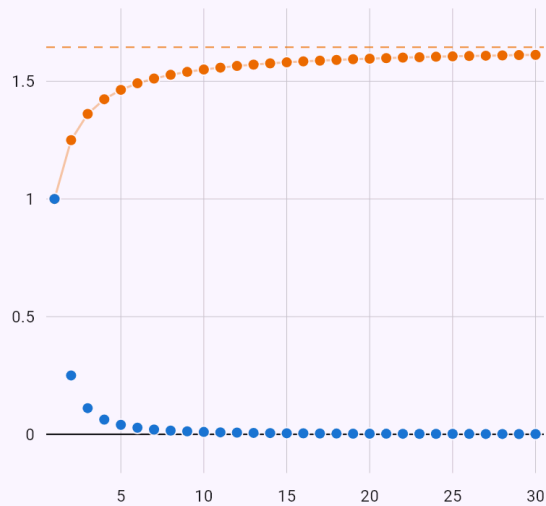
The series converges.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449$$

Steps

- The necessary condition for convergence holds (it doesn't decide by itself): $\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$
- The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is $\sum 1/n^p$ with $p = 2 > 1$ – it converges.
- A known series' sum: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Graph



● terms a_n ● partial sums s_n — the sum

The points are the terms a_n and the partial sums $s_n = a_1 + \dots + a_n$. If the series converges, the terms approach zero and the partial sums its sum.

The alternating series test – converges conditionally.

Input: `sum((-1)^(n+1)/n)`

Result

The series converges, but not absolutely.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2 \approx 0.6931$$

Steps

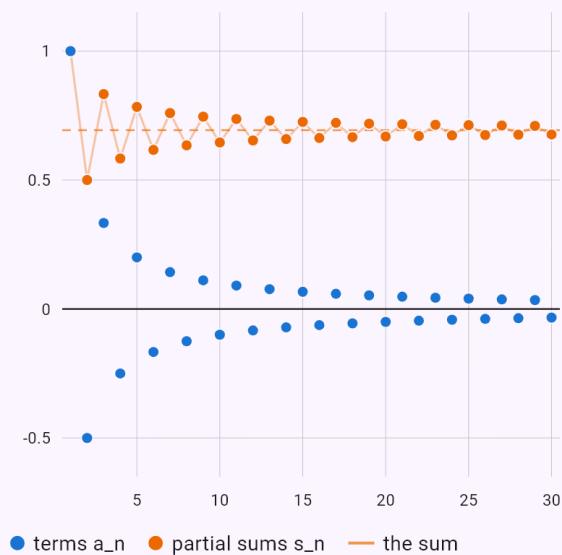
- The necessary condition for convergence holds (it doesn't decide by itself): $\lim_{n \rightarrow \infty} \frac{(-1)^{n+1}}{n} = 0$
- The terms have both signs – absolute convergence first

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$
- The harmonic series diverges – the terms grouped by powers of two, each group at least $1/2$

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4} \right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \dots > 1 + \frac{1}{2}$$
- Leibniz's test: $|a_n| = \frac{1}{n}$ decreases and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ – the series converges.
- Conclusion: the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, but not absolutely (by Leibniz's test).

• A known series' sum: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

Graph



The points are the terms a_n and the partial sums $s_n = a_1 + \dots + a_n$. If the series converges, the terms approach zero and the partial sums its sum.

The ratio test and the sum.

Input: `sum(n/2^n)`

Result

The series converges.

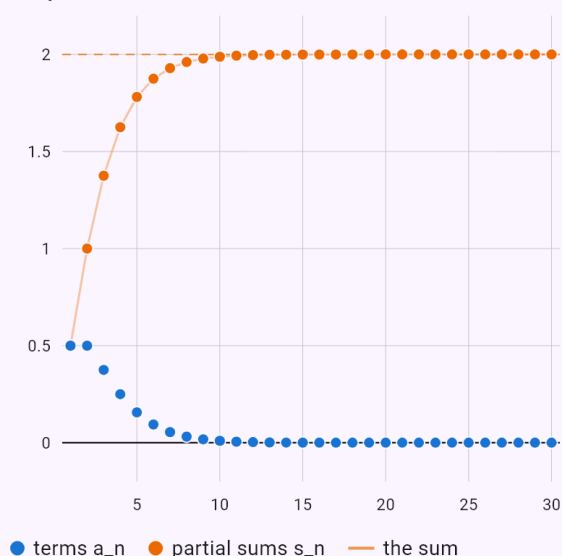
$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$$

Steps

- The necessary condition for convergence holds (it doesn't decide by itself): $\lim_{n \rightarrow \infty} \frac{n}{2^n} = 0$
- The ratio (d'Alembert's) test – the series converges.

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n+1}{2n} = \frac{1}{2} < 1$$
- A series $\sum P(n) \cdot q^n$ summed by differentiating the geometric series: $\sum_{n=0}^{\infty} n q^n = \frac{q}{(1-q)^2}$, $s = 2$

Graph



The points are the terms a_n and the partial sums $s_n = a_1 + \dots + a_n$. If the series converges, the terms approach zero and the partial sums its sum.

The necessary condition fails – the series diverges.

Input: `sum(n/(n+1))`

Result

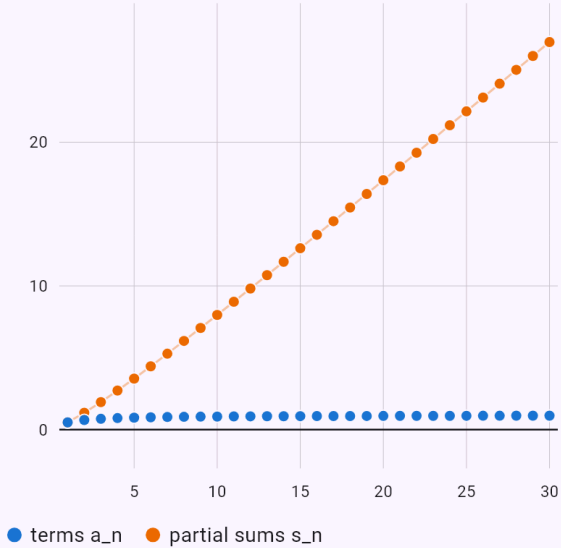
The series diverges.

Steps

- The necessary condition for convergence fails:

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$$

Graph



The points are the terms a_n and the partial sums $s_n = a_1 + \dots + a_n$. If the series converges, the terms approach zero and the partial sums its sum.

A telescoping sum – the first and the last term remain.

Input: `sum(sqrt(2i+1) - sqrt(2i-1), i, 1, 60)`

Result

$$\sum_{i=1}^{60} (\sqrt{2i+1} - \sqrt{2i-1}) = 10$$

Steps

- A telescoping sum – neighbouring terms cancel, only the first and the last remain:

$$\sum_{i=1}^{60} (\sqrt{2i+1} - \sqrt{2i-1}) = 11 - 1 = 10$$

An equation with the sum of a geometric series.

Input: $\text{sum}((2^x)^n, n, 1, \infty) = 1$

Result

$$x = -1$$

Steps

- The series is geometric with the ratio $q = 2^x$ and converges for $|2^x| < 1$; its sum:

$$\sum_{n=1}^{\infty} (2^x)^n = \frac{a_1}{1-q} = \frac{2^x}{1-2^x}$$

- Solve: $\frac{2^x}{1-2^x} = 1 \Rightarrow x = -1$
- $x = -1$ meets the condition of convergence: $\left|\frac{1}{2}\right| < 1$

Power series

For a power series the app finds the radius of convergence by the ratio test, examines the endpoints as numerical series and states the interval of convergence; where possible also the sum. The graph shows the partial sums with a slider for the number of terms.

Examples

The interval of convergence $[-1, 1)$.

Input: `sum(x^n/n)`

Result

$$x \in [-1, 1)$$

Steps

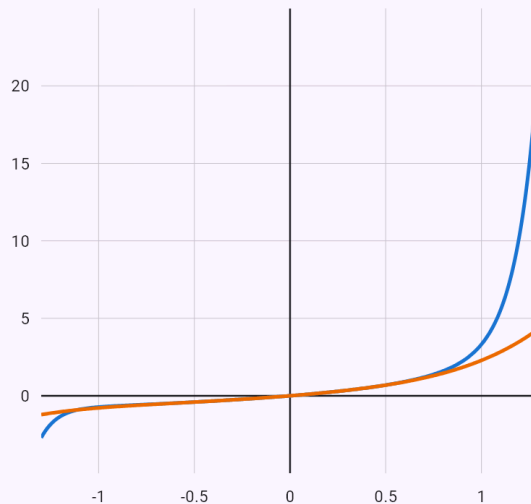
- The radius of convergence $R = 1$ by the ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \right| = |x| < 1$$

- The endpoint $x = -1$: the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, but not absolutely (by Leibniz's test).
- The endpoint $x = 1$: the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by grouping the harmonic series' terms.
- The interval of convergence: $x \in [-1, 1)$

Graph

Terms: 5



$$s_{15}(x) = \frac{x^{15}}{15} + \frac{x^{14}}{14} + \frac{x^{13}}{13} + \frac{x^{12}}{12} + \frac{x^{11}}{11} + \frac{x^{10}}{10}$$
$$s_5(x) = \frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x$$

Pick the number of terms with the slider – the partial sums approach the series' sum only inside the interval of convergence.

Taylor series

The app derives the Taylor series of a function from known expansions (e^x , $\sin x$, $\ln(1+x)$, ...) by substitution and determines its interval of convergence.

How to enter it

`taylor(e^x, 0, oo)`

The function, the point and oo instead of the degree; or the “Taylor polynomial” chip → “Series”.

Examples

The series of $\ln(1 + x)$ and its interval of convergence.

Input: `taylor(ln(1 + x), 0, oo)`

Result

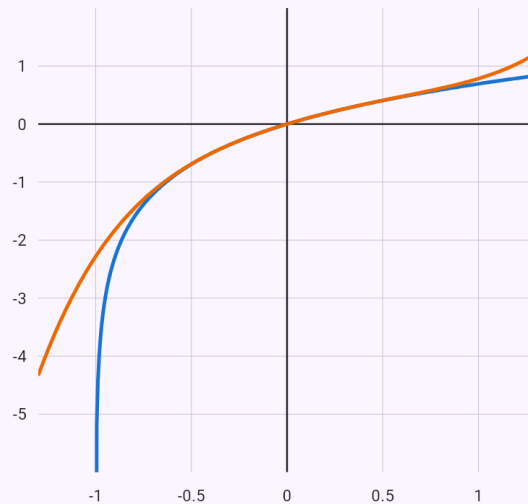
$$\ln(x + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad x \in (-$$

Steps

- From the known expansion $\ln(1 + u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} u^n}{n}$ with $u = x$: $\ln(x + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$
- The radius of convergence $R = 1$ by the ratio test
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| -\frac{xn}{n+1} \right| = |x| < 1$$
- The endpoint $x = -1$: the series $\sum_{n=1}^{\infty} -\frac{1}{n}$ diverges by grouping the harmonic series' terms.
- The endpoint $x = 1$: the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, but not absolutely (by Leibniz's test).
- The interval of convergence: $x \in (-1, 1]$

Graph

Terms: 5



$$s(x) = \ln(x + 1)$$

$$s_5(x) = \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x$$

Pick the number of terms with the slider – the partial sums approach the series' sum only inside the interval of convergence.

Fourier series

The app builds the Fourier series of a function on an interval taken as one period (without one on $[-\pi, \pi]$), or the sine or cosine series of a function on $[0, L]$. The function can also be given piecewise.

It computes the coefficients by integrals with the full working (integration by parts with the parameter n), substitutes $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ and uses symmetry for even and odd functions. At discontinuities it determines the sum of the series by Dirichlet's theorem, and by substituting a point and by Parseval's identity it derives sums of numerical series (such as $\sum 1/n^2 = \pi^2/6$). The graph shows the periodic extension of the function and the partial sums with a slider.

How to enter it

<code>fourier(x^2, -pi, pi)</code>	The function and the interval.
<code>fouriersin(x, 0, pi)</code>	The sine series (fouriercos – the cosine series).
<code>-1, -pi < x < 0</code> <code>1, 0 < x < pi</code>	A piecewise function line by line, with the “Fourier series” chip.
<code>x^2</code>	A function with the “Fourier series” chip – choose the kind of series and the interval there.

Examples

An even function – cosines only; the sums $\sum 1/n^2$ and $\sum 1/n^4$.

Input: `fourier(x^2)`

Result

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx), \quad ;$$

Steps

- The interval $[-\pi, \pi]$ is one period: $T = 2\pi$, half the period $l = \pi$. We look for the series with the coefficients

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

- The function is even, $f(-x) = x^2 = f(x)$, so the sine coefficients are zero and the others can be computed over half the interval

$$b_n = 0$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$$

- The coefficient a_0 : $a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx$
- From the table of basic integrals: $\int x^2 dx = \frac{x^3}{3}$
- Substituting the limits: $a_0 = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}$
- The coefficient a_n : $a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) dx$
- By parts ($\int u \cdot v' = u \cdot v - \int u' \cdot v$), $u = x^2$, $v' = \cos(nx)$, so $u' = 2x$, $v = \frac{\sin(nx)}{n}$
$$\int x^2 \cos(nx) dx = \frac{x^2 \sin(nx)}{n} - \int \frac{2x \sin(nx)}{n} dx$$
- The constant taken out of the integral:
$$\int \frac{2x \sin(nx)}{n} dx = \frac{2}{n} \int x \sin(nx) dx$$
- By parts ($\int u \cdot v' = u \cdot v - \int u' \cdot v$), $u = x$, $v' = \sin(nx)$, so $u' = 1$, $v = -\frac{\cos(nx)}{n}$
$$\int x \sin(nx) dx = -\frac{x \cos(nx)}{n} - \int -\frac{\cos(nx)}{n} dx$$
- The constant taken out of the integral:
$$\int -\frac{\cos(nx)}{n} dx = -\frac{1}{n} \int \cos(nx) dx$$
- From the table – the inner function is linear ($ax + b$), so the result is divided by a : $\int \cos(nx) dx = \frac{\sin(nx)}{n}$

- For a whole number n , $\sin n\pi = 0$, $\cos n\pi = (-1)^n$

- Substituting the limits

$$a_n = \frac{2}{\pi} \left[\frac{x^2 \sin(nx)}{n} + \frac{2x \cos(nx)}{n^2} - \frac{2 \sin(nx)}{n^3} \right]_0^\pi = -$$

- The coefficients substituted into the series

$$s(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$$

- The periodic extension of the function is continuous, so by Dirichlet's theorem the sum of the series equals $f(x)$ for $x \in [-\pi, \pi]$

- Substituting $x = 0$ gives the sum of a series of numbers

$$s(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

- Substituting $x = \pi$ gives the sum of a series of numbers

$$s(\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} = \pi^2 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

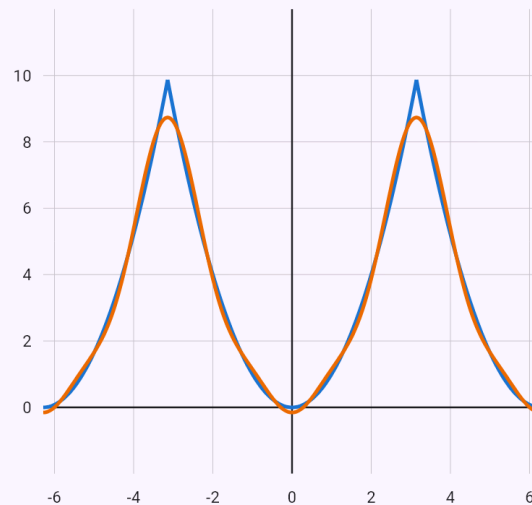
- From Parseval's identity

$$\frac{2}{\pi} \int_0^\pi f^2(x) dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Graph

Terms up to $n = 3$



— $f(x)$
— $s_3(x)$

The function is extended periodically. The slider adds terms of the series – the partial sums approach the function; at a jump they overshoot (the Gibbs phenomenon) and pass through the average of both sides, marked by a point.

A piecewise function – jumps and the Gibbs phenomenon.

Input: $-1, -\pi < x < 0$

Result

$$f(x) = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin((2k-1)x)$$

Steps

- The interval $[-\pi, \pi]$ is one period: $T = 2\pi$, half the period $l = \pi$. We look for the series with the coefficients

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

- The function is odd, $f(-x) = -f(x)$, so the constant term and the cosine coefficients are zero and the sine coefficients can be computed over half the interval

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

- The coefficient b_n : $b_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx) dx$

- From the table – the inner function is linear ($ax + b$), so the result is divided by a : $\int \sin(nx) dx = -\frac{\cos(nx)}{n}$

- For a whole number n , $\sin n\pi = 0$, $\cos n\pi = (-1)^n$

- Substituting the limits

$$b_n = \frac{2}{\pi} \left[-\frac{\cos(nx)}{n} \right]_0^{\pi} = \frac{2 - 2(-1)^n}{n\pi}$$

- The coefficients substituted into the series:

$$s(x) = \sum_{n=1}^{\infty} \frac{2 - 2(-1)^n}{n\pi} \sin(nx)$$

- The terms with an even n are zero – only the odd $n = 2k - 1$ remain

$$s(x) = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin((2k-1)x)$$

- By Dirichlet's theorem the sum of the series equals $f(x)$ for $x \in (-\pi, 0) \cup (0, \pi)$; at a point of discontinuity it is the average of the one-sided limits

$$s(-\pi) = s(\pi) = \frac{f(\pi^-) + f(-\pi^+)}{2} = \frac{1 + (-1)}{2} = 0$$

$$s(0) = \frac{f(0^-) + f(0^+)}{2} = \frac{-1 + 1}{2} = 0$$

- Substituting $x = \frac{\pi}{2}$ gives the sum of a series of numbers

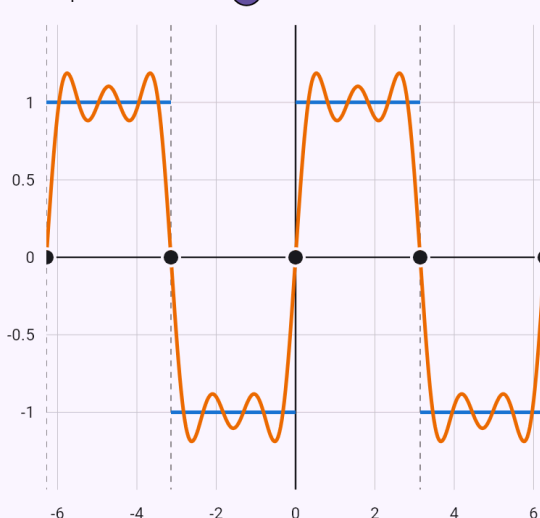
$$s\left(\frac{\pi}{2}\right) = \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{\pi(2k-1)} = 1 \Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \frac{\pi}{4}$$

- From Parseval's identity

$$\frac{2}{\pi} \int_0^{\pi} f^2(x) dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$2 = \sum_{k=1}^{\infty} \frac{16}{\pi^2(2k-1)^2} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

Graph

Terms up to $n = 5$ 

— $f(x)$
— $s_5(x)$

The function is extended periodically. The slider adds terms of the series – the partial sums approach the function; at a jump they overshoot (the Gibbs phenomenon) and pass through the average of both sides, marked by a point.

A sine series and the sum $1 - 1/3^3 + 1/5^3 - \dots = \pi^3/32$.

Input: `fouriersin(x*(pi - x), 0, pi)`

Result

$$f(x) = \sum_{k=1}^{\infty} \frac{8}{\pi(2k-1)^3} \sin((2k-1)x)$$

Steps

- We extend the function from $[0, \pi]$ as an odd function, then periodically ($T = 2\pi$) – a sine series

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx), \text{ where}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

- The coefficient b_n : $b_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin(nx) dx$
- By parts ($\int u \cdot v' = u \cdot v - \int u' \cdot v$), $u = x(\pi - x)$, $v' = \sin(nx)$, so $u' = \pi - 2x$, $v = -\frac{\cos(nx)}{n}$

$$\int x(\pi - x) \sin(nx) dx = \frac{x(x - \pi) \cos(nx)}{n} - \int \frac{(2x - \pi) \cos(nx)}{n} dx$$

- The constant taken out of the integral

$$\int \frac{(2x - \pi) \cos(nx)}{n} dx = \frac{1}{n} \int (2x - \pi) \cos(nx) dx$$

- By parts ($\int u \cdot v' = u \cdot v - \int u' \cdot v$), $u = 2x - \pi$, $v' = \cos(nx)$, so $u' = 2$, $v = \frac{\sin(nx)}{n}$

$$\int (2x - \pi) \cos(nx) dx = \frac{(2x - \pi) \sin(nx)}{n} - \int \frac{2 \sin(nx)}{n} dx$$

- The constant taken out of the integral:
 $\int \frac{2 \sin(nx)}{n} dx = \frac{2}{n} \int \sin(nx) dx$
- From the table – the inner function is linear ($ax + b$), so the result is divided by a : $\int \sin(nx) dx = -\frac{\cos(nx)}{n}$
- For a whole number n , $\sin n\pi = 0$, $\cos n\pi = (-1)^n$
- Substituting the limits

$$b_n = \frac{2}{\pi} \left[\frac{x^2 \cos(nx)}{n} - \frac{\pi x \cos(nx)}{n} + \frac{(\pi - 2x) \sin(nx)}{n^2} \right]$$

- The coefficients substituted into the series:

$$s(x) = \sum_{n=1}^{\infty} \frac{4 - 4(-1)^n}{n^3 \pi} \sin(nx)$$

- The terms with an even n are zero – only the odd $n = 2k - 1$ remain

$$s(x) = \sum_{k=1}^{\infty} \frac{8}{\pi(2k-1)^3} \sin((2k-1)x)$$

- The periodic extension of the function is continuous, so by Dirichlet's theorem the sum of the series equals $f(x)$ for $x \in [0, \pi]$
- Substituting $x = \frac{\pi}{2}$ gives the sum of a series of numbers

$$s\left(\frac{\pi}{2}\right) = \sum_{k=1}^{\infty} \frac{8(-1)^{k+1}}{\pi(2k-1)^3} = \frac{\pi^2}{4} \Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}$$

• From Parseval's identity $\frac{2}{\pi} \int_0^{\pi} f^2(x) dx = \sum_{n=1}^{\infty} b_n^2$

$$\frac{\pi^4}{15} = \sum_{k=1}^{\infty} \frac{64}{\pi^2(2k-1)^6} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^6} = \frac{\pi^6}{960}$$

Graph

Terms up to $n = 5$



— $f(x)$
— $s_5(x)$

The function is extended periodically. The slider adds terms of the series – the partial sums approach the function; at a jump they overshoot (the Gibbs phenomenon) and pass through the average of both sides, marked by a point.

Known limitations

- An unbounded function (such as $\ln x$ on $[0, 2\pi]$) cannot be expanded – the series would diverge at the point where the function grows without bound.
- When the coefficients lead to an integral without an elementary antiderivative ($\ln x$ on $[1, 3]$), they are expressed with the sine and cosine integrals.

Differential equations

The app solves first-order ordinary differential equations: separable, linear, exact, homogeneous ($y' = f(y/x)$), Bernoulli equations and equations $y' = f(ax + by + c)$. It recognizes the type itself and shows the steps as taught at school – separating the variables, variation of the constant or an integrating factor, a substitution, the potential of an exact equation. The integrals in the steps have steps of their own.

Higher orders: linear equations with constant coefficients of any order by the characteristic equation (multiple and complex roots too), a particular solution by the special right side (resonance and a sum of right sides too) or by variation of constants, an initial value problem at zero by the Laplace transform too (the "Method" chip). Further Euler's equation by $x = e^t$, $y^{(n)} = f(x)$ by repeated integration and the reduction of order – when y is missing ($p = y'$) or x is ($y' = p(y)$).

The equation $y' = y \cdot (a(x) \cdot \ln y + b(x))$ becomes linear by the substitution $z = \ln y$. A linear second-order equation with variable coefficients is solved when one solution can be guessed (x^k , e^{cx} , ...): the second one by reducing the order (d'Alembert), the right side by variation of constants. The order is reduced by more than one too ($p = y^{(m)}$ with a second-order equation for p). With parameters the cases are told apart – a damped oscillator by the sign of $\delta^2 - \omega^2$ (overdamped, critical, underdamped), a resonance ($y'' + \omega^2 y = \cos 2t$ for $\omega = 2$) and the sign of a constant the integral depends on ($y'' = 2yy'$). The Laplace transform also solves an initial value problem at a point other than 0 (by the shift $\tau = t - t_0$) and with parameters.

The result is the general solution with a constant C , for higher orders with constants $C_1, C_2, \dots$ (explicitly, or implicitly where it can't be explicit); with conditions, the solution that satisfies them – initial and boundary ones, including "no solution" and "infinitely many solutions". Constant solutions lost by dividing are found too. Every solution is checked by substituting it into the equation. For the first order the graph shows the direction field and solutions for different C , for higher orders solutions for different constants; for an autonomous second-order equation (no x) a switch shows the phase plane (y, y') too.

How to enter it

$y' = x*y$	The derivative with a prime.
$dy/dx = x \ y^2$	Leibniz's notation – it names the variable too (dm/dt , dT/dt).
$\frac{dy}{dx} = \frac{y}{x} + x$	Notation from a photo.
$(2xy + 6x)dx + (x^2 + 4y^3)dy = 0$	The form $M dx + N dy = 0$ (an exact equation).
$y' = y^2 + 1, y(0) = 1$	An initial condition after a comma or on the next line.

$$dm/dt = m + 2, m(0) = 0$$

The function and the variable can be any letters.

$$y''' - 3y' + 2y = 3e^{(2x)}$$

A higher-order equation – the derivatives with more primes.

$$y^{(4)} + 2y'' + y = 0$$

Higher derivatives as $y^{(4)}$ (from a photo too).

$$y'' + y = 0, y(0) = 1, y'(0) = 2$$

Initial conditions $y(0), y'(0)$ – or boundary ones at different points.

$$\ddot{x} + \omega^2 x = 0$$

Newton's dots (derivatives by time t) and Greek parameters.

Examples

A separable equation – the general solution and the direction field.

Input: $y' = x*y$

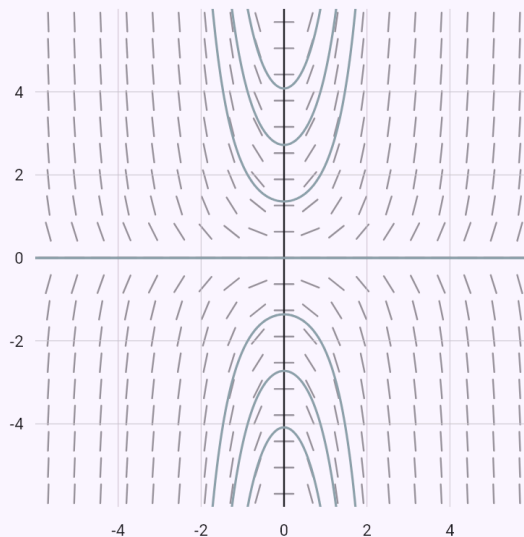
Result

$$y = Ce^{\frac{x^2}{2}}, \quad C \in \mathbb{R}$$

Steps

- A separable equation – the right side is a product of a function of the variable and a function of the unknown:
 $y' = xy$
- Separating the variables (dividing, assuming $y \neq 0$):
 $\frac{dy}{y} = x dx$
- Integrating both sides: $\int \frac{dy}{y} = \int x dx$
- From the table of basic integrals: $\int \frac{1}{y} dy = \ln |y|$
- From the table of basic integrals: $\int x dx = \frac{x^2}{2}$
- The general solution in implicit form: $\ln |y| = \frac{x^2}{2} + C$
- Exponentiating – the constant $\pm e^C$ is called C again:
 $y = Ce^{\frac{x^2}{2}}$
- The constant solution $y = 0$, excluded by dividing, is in the general solution for $C = 0$
- Check – the solution's derivative $y' = Cxe^{\frac{x^2}{2}}$ substituted into the equation gives an identity.

Graph



— solutions for different C

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

The initial condition determines the constant C.

Input: $y' = y^2 + 1$, $y(0) = 1$

Result

$$y = \tan\left(x + \frac{\pi}{4}\right)$$

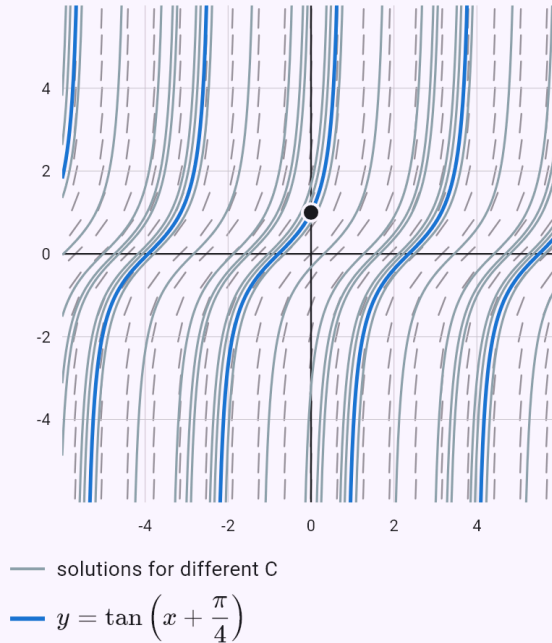
Steps

- A separable equation – the right side is a product of a function of the variable and a function of the unknown:
 $y' = y^2 + 1$
- Separating the variables (dividing, assuming $y^2 + 1 \neq 0$):
 $\frac{dy}{y^2 + 1} = dx$
- Integrating both sides: $\int \frac{dy}{y^2 + 1} = \int dx$
- From the table of basic integrals:
 $\int \frac{1}{y^2 + 1} dy = \arctan y$
- From the table of basic integrals: $\int 1 dx = x$
- The general solution in implicit form: $\arctan y = x + C$
- Expressing the solution: $y = \tan(x + C)$
- Substituting the initial condition $y(0) = 1$:
 $1 = \tan C \Rightarrow C = \frac{\pi}{4}$
- The solution of the initial condition (the particular solution): $y = \tan\left(x + \frac{\pi}{4}\right)$

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere. Highlighted is the solution of the initial condition; the point is the initial condition.

- Check – the solution's derivative $y' = \frac{1}{\cos^2\left(x + \frac{\pi}{4}\right)}$ substituted into the equation gives an identity.

Graph



A linear equation by variation of the constant.

Input: $y' + 2y = e^x$ Variation of the constant

Result

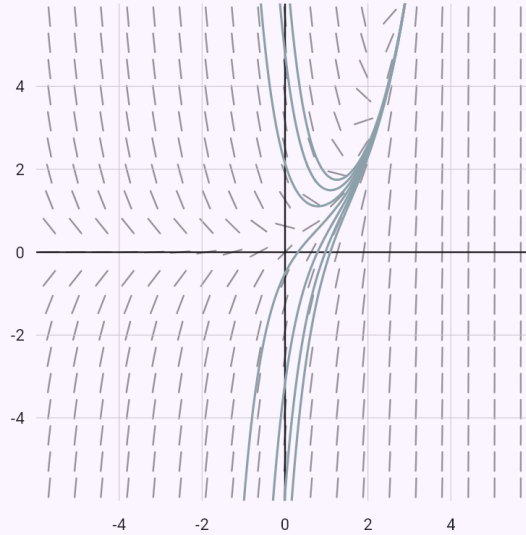
$$y = Ce^{-2x} + \frac{e^x}{3}, \quad C \in \mathbb{R}$$

Steps

- A first-order linear equation $y' + p(x) \cdot y = q(x)$:
 $y' + 2y = e^x$
- First the homogeneous equation (zero right side):
 $y' + 2y = 0$
- From the table of basic integrals: $\int 2 dx = 2x$
- The homogeneous equation's solution: $y_h = Ce^{-2x}$
- Variation of the constant – we look for the solution as $y = C(x)e^{-2x}$; substituted into the equation:
 $C'(x)e^{-2x} = e^x$
- Hence $C'(x) = e^{3x}$
- From the table – the inner function is linear ($ax + b$), so the result is divided by a : $\int e^{3x} dx = \frac{e^{3x}}{3}$
- Integrating: $C(x) = \int e^{3x} dx = \frac{e^{3x}}{3} + C$
- The general solution: $y = Ce^{-2x} + \frac{e^x}{3}$

- Check – the solution's derivative $y' = \frac{e^x}{3} - 2Ce^{-2x}$ substituted into the equation gives an identity.

Graph



— solutions for different C

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

The same equation by an integrating factor.

Input: $y' + 2y = e^x$ Integrating factor

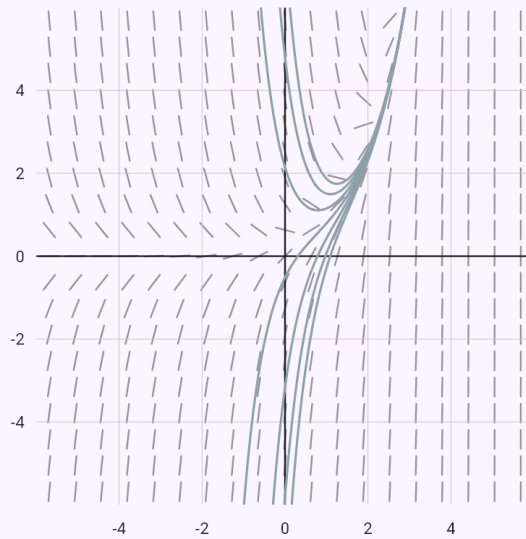
Result

$$y = Ce^{-2x} + \frac{e^x}{3}, \quad C \in \mathbb{R}$$

Steps

- A first-order linear equation $y' + p(x) \cdot y = q(x)$:
 $y' + 2y = e^x$
- From the table of basic integrals: $\int 2 dx = 2x$
- The integrating factor: $\mu(x) = e^{\int 2 dx} = e^{2x}$
- Multiplying by the integrating factor – the left side is the derivative of a product: $(ye^{2x})' = e^{3x}$
- From the table – the inner function is linear ($ax + b$), so the result is divided by a : $\int e^{3x} dx = \frac{e^{3x}}{3}$
- Integrating both sides: $ye^{2x} = \frac{e^{3x}}{3} + C$
- The general solution: $y = Ce^{-2x} + \frac{e^x}{3}$
- Check – the solution's derivative $y' = \frac{e^x}{3} - 2Ce^{-2x}$ substituted into the equation gives an identity.

Graph



— solutions for different C

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

A homogeneous equation – the substitution $u = y/x$.

Input: $y' = (x - y)/(x + y)$

Result

$$x^2 - 2xy - y^2 = C, \quad C \in \mathbb{R}$$

Steps

- A homogeneous equation – the right side depends only on the ratio y/x : $y' = \frac{x - y}{x + y}$
- The substitution $u = \frac{y}{x}$, $y = ux$, $y' = u + xu'$ turns the equation into $xu' = -u - 1 + \frac{2}{u+1}$
- Separating the variables (dividing, assuming $u^2 + 2u - 1 \neq 0$): $\frac{u+1}{u^2+2u-1} du = -\frac{1}{x} dx$
- Integrating both sides: $\int \frac{u+1}{u^2+2u-1} du = \int -\frac{1}{x} dx$
- The constant taken out of the integral:

$$\int -\frac{u+1}{-u^2-2u+1} du = -\int \frac{u+1}{-u^2-2u+1} du$$
- The numerator is the derivative of the denominator (or its multiple), $f(x) = -u^2 - 2u + 1$, by the formula

$$\int \frac{f'(u)}{f(u)} du = \ln |f(u)| + C$$

$$\int \frac{u+1}{-u^2-2u+1} du = -\frac{1}{2} \int \frac{-2u-2}{-u^2-2u+1} du = -\frac{1}{2} \ln$$

- The constant taken out of the integral:

$$\int -\frac{1}{x} dx = -\int \frac{1}{x} dx$$

- From the table of basic integrals: $\int \frac{1}{x} dx = \ln |x|$

- The general solution in implicit form:

$$\frac{\ln |-u^2 - 2u + 1|}{2} = C - \ln |x|$$

- Exponentiating – the constant $\pm e^C$ is called C again:

$$-u^2 - 2u + 1 = \frac{C}{x^2}$$

- The constant solution $u = -1 - \sqrt{2}$, excluded by dividing, is in the general solution for $C = 0$

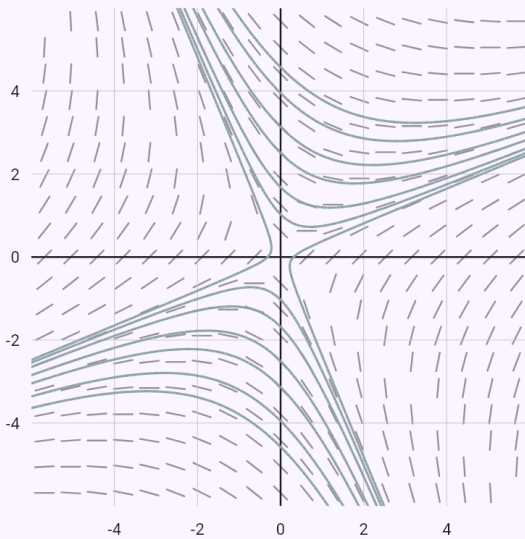
- The constant solution $u = -1 + \sqrt{2}$, excluded by dividing, is in the general solution for $C = 0$

- Back to the original unknown: $\frac{x^2 - 2xy - y^2}{x^2} = \frac{C}{x^2}$

- The general solution in implicit form:

$$x^2 - 2xy - y^2 = C$$

Graph



— solutions for different C

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

A Bernoulli equation – the substitution $z = y^{1-n}$ and the constant solution $y = 0$.

Input: $xy' - y = -xy^2$

Result

$$y = \frac{2x}{x^2 + C}, \quad C \in \mathbb{R}; \quad y = 0$$

Steps

• A Bernoulli equation $y' + p(x) \cdot y = q(x) \cdot y^n$: $y' - \frac{y}{x} = -y^2$

• The substitution $z = \frac{1}{y}$, $z' = -\frac{1}{y^2} y'$ turns the equation into $z' + \frac{z}{x} = 1$

• First the homogeneous equation (zero right side): $z' + \frac{z}{x} = 0$

• From the table of basic integrals: $\int \frac{1}{x} dx = \ln|x|$

• The homogeneous equation's solution: $z_h = \frac{C}{x}$

• Variation of the constant – we look for the solution as

$$z = C(x) \cdot \frac{1}{x}; \text{ substituted into the equation:}$$

$$C'(x) \cdot \frac{1}{x} = 1$$

• Hence $C'(x) = x$

• From the table of basic integrals: $\int x dx = \frac{x^2}{2}$

• Integrating: $C(x) = \int x dx = \frac{x^2}{2} + C$

• The general solution: $z = \frac{x}{2} + \frac{C}{x}$

• Back to the original unknown: $\frac{1}{y} = \frac{x}{2} + \frac{C}{x}$

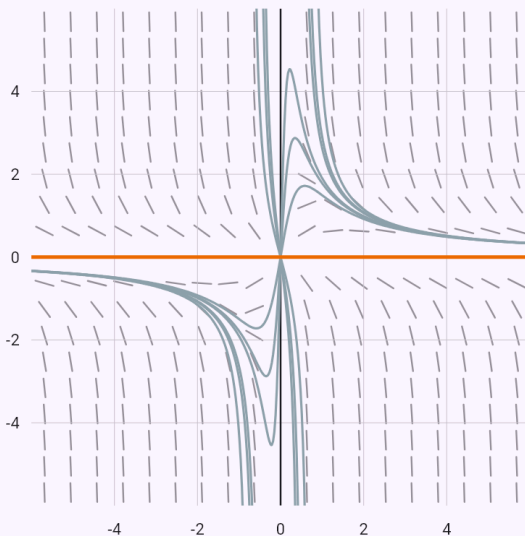
• Expressing the solution: $y = \frac{2x}{x^2 + 2C}$

• Dividing excluded the constant solution $y = 0$ – it is another solution of the equation.

• Renaming the constant (a multiple of an arbitrary constant is an arbitrary constant): $y = \frac{2x}{x^2 + C}$

• Check – the solution's derivative $y' = \frac{2C - 2x^2}{x^4 + 2Cx^2 + C^2}$ substituted into the equation gives an identity.

Graph



— solutions for different C

— $y = 0$

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

An exact equation – the potential $F(x, y) = C$.

Input: $(2xy + 6x)dx + (x^2 + 4y^3)dy = 0$

Result

$$y^4 + x^2y + 3x^2 = C, \quad C \in \mathbb{R}$$

Steps

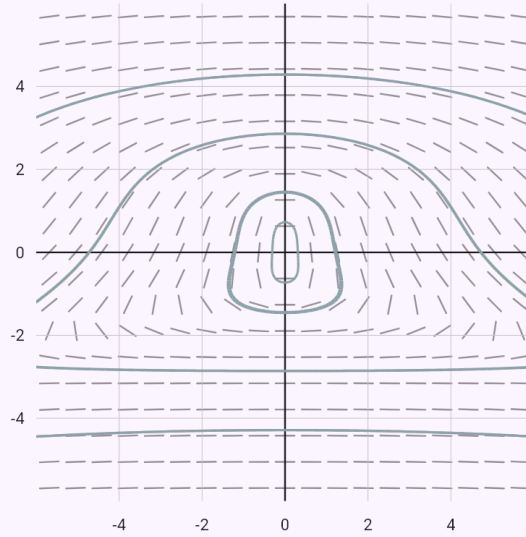
- An equation $M dx + N dy = 0$ – we check whether it is exact (the left side a total differential):
 $(2xy + 6x)dx + (4y^3 + x^2)dy = 0$
- $M = 2xy + 6x$, $N = 4y^3 + x^2$; the equation is exact, since
 $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$
- The integral of a sum is the sum of the integrals:
 $\int (6x + 2yx) dx = \int 6x dx + \int 2yx dx$
- The constant taken out of the integral:
 $\int 6x dx = 6 \int x dx$
- From the table of basic integrals: $\int x dx = \frac{x^2}{2}$
- The constant taken out of the integral:
 $\int 2yx dx = 2y \int x dx$
- From the table of basic integrals: $\int x dx = \frac{x^2}{2}$
- The potential F by integrating M with respect to x :
 $F(x, y) = \int (2xy + 6x) dx = x^2y + 3x^2 + \varphi(y)$

- The constant taken out of the integral:

$$\int 4y^3 dy = 4 \int y^3 dy$$

- From the table of basic integrals: $\int y^3 dy = \frac{y^4}{4}$
- From $\partial F / \partial y = N$: $\varphi'(y) = 4y^3 \Rightarrow \varphi(y) = y^4$
- The general solution in implicit form:
 $y^4 + x^2y + 3x^2 = C$

Graph



— solutions for different C

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

The substitution $z = x - y$; Symja would lose a factor 2 here, the app checks the result.

Input: $y' = \cos(x - y)$, $y(0) = \pi/2$

Result

$$y = x + 2 \operatorname{arccot}(x + 1)$$

Steps

- The right side is a function of $ax + by + c$ – we take it as a new unknown: $y' = \cos(x - y)$

- The substitution $z = x - y$, $z' = 1 - y'$ turns the equation into $z' = 1 - \cos z$

- Separating the variables (dividing, assuming $1 - \cos z \neq 0$): $\frac{dz}{1 - \cos z} = dx$

- Integrating both sides: $\int \frac{dz}{1 - \cos z} = \int dx$

- No simpler substitution fits – the universal substitution $t = \tan\left(\frac{z}{2}\right)$

$$\sin z = \frac{2t}{1+t^2}, \cos z = \frac{1-t^2}{1+t^2}, dz = \frac{2dt}{1+t^2}$$

$$\int \frac{1}{1 - \cos z} dz = \int \frac{1}{t^2} dt$$

- From the table of basic integrals: $\int \frac{1}{t^2} dt = -\frac{1}{t}$

- Back to the original variable: $t = \tan\left(\frac{z}{2}\right) : -\cot\left(\frac{z}{2}\right)$

- From the table of basic integrals: $\int 1 dx = x$

- The general solution in implicit form: $-\cot\left(\frac{z}{2}\right) = x + C$

- Expressing the solution: $z = 2 \operatorname{arccot}(C - x)$

- Dividing excluded the constant solution $z = 0$ – it is another solution of the equation.

- Back to the original unknown: $y = x - 2 \operatorname{arccot}(C - x)$

- Renaming the constant (a multiple of an arbitrary constant is an arbitrary constant): $y = x + 2 \operatorname{arccot}(x + C)$

- Substituting the initial condition $y(0) = \frac{\pi}{2}$: $\frac{\pi}{2} = 2 \operatorname{arccot} C \Rightarrow C = 1$

- The solution of the initial condition (the particular solution): $y = x + 2 \operatorname{arccot}(x + 1)$

- Check – the solution's derivative $y' = \frac{x^2 + 2x}{x^2 + 2x + 2}$ substituted into the equation gives an identity.

The characteristic equation with complex roots $-2 \pm 3i$.

Input: $y'' + 4y' + 13y = 0$

Result

$$y = e^{-2x}(C_1 \cos(3x) + C_2 \sin(3x)),$$

Steps

- A linear equation with constant coefficients: $y'' + 4y' + 13y = 0$

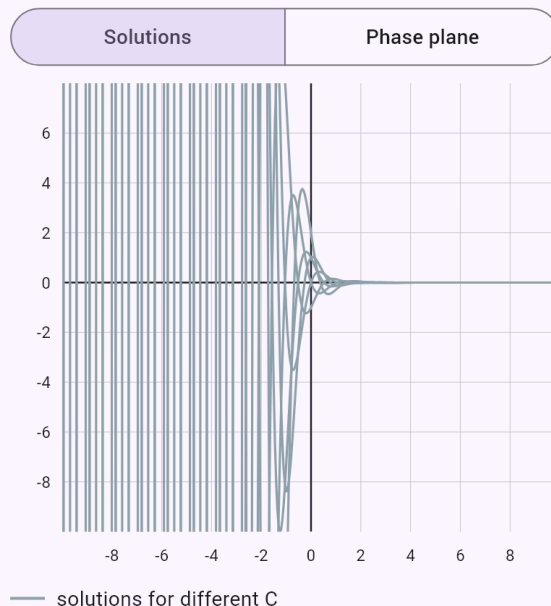
- We look for solutions $y = e^{\lambda x}$; substituting and dividing by $e^{\lambda x}$ gives the characteristic equation: $\lambda^2 + 4\lambda + 13 = 0$

- The discriminant $D = 4^2 - 4 \cdot 13 = -36$; the roots: $\lambda_{1,2} = -2 \pm 3i$

- Each root λ gives the solution $e^{\lambda x}$, a root of multiplicity k also $x \cdot e^{\lambda x}$, ..., $x^{k-1} \cdot e^{\lambda x}$, and a pair $\alpha \pm \beta i$ the solutions $e^{\alpha x} \cdot \cos \beta x$ and $e^{\alpha x} \cdot \sin \beta x$. The fundamental system: $y_1 = e^{-2x} \cos(3x)$, $y_2 = e^{-2x} \sin(3x)$

- The general solution: $y = e^{-2x}(C_1 \cos(3x) + C_2 \sin(3x))$

Graph



The grey curves are solutions for various values of the constants.

A special right side with resonance – 2 is a root of the characteristic equation.

Input: $y'' - 3y' + 2y = 3e^{2x}$

Special right side

Result

$$y = C_1 e^x + C_2 e^{2x} + 3xe^{2x}, \quad C_1, C_2 \in \mathbb{R}$$

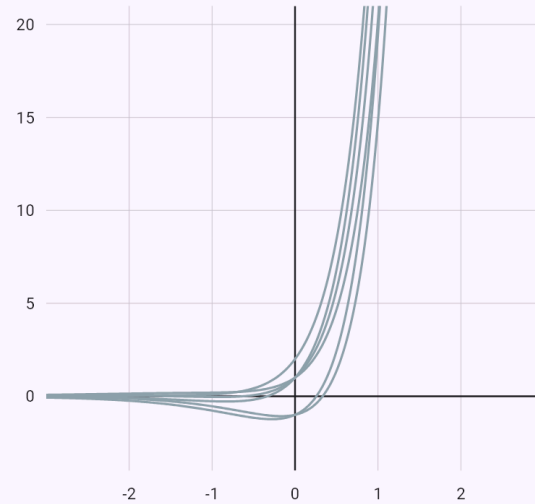
Steps

- A linear equation with constant coefficients:
 $y'' - 3y' + 2y = 3e^{2x}$
- First the homogeneous equation (zero right side):
 $y'' - 3y' + 2y = 0$
- We look for solutions $y = e^{(\lambda x)}$; substituting and dividing by $e^{(\lambda x)}$ gives the characteristic equation:
 $\lambda^2 - 3\lambda + 2 = 0$
- The discriminant $D = (-3)^2 - 4 \cdot 2 = 1$; the roots:
 $\lambda_1 = 1, \lambda_2 = 2$
- Each root λ gives the solution $e^{(\lambda x)}$, a root of multiplicity k also $x \cdot e^{(\lambda x)}, \dots, x^{(k-1)} \cdot e^{(\lambda x)}$, and a pair $\alpha \pm \beta i$ the solutions $e^{(\alpha x)} \cdot \cos \beta x$ and $e^{(\alpha x)} \cdot \sin \beta x$. The fundamental system: $y_1 = e^x, y_2 = e^{2x}$
- The homogeneous equation's solution:
 $y_h = C_1 e^x + C_2 e^{2x}$
- The right side $3e^{2x}$ has a special form; the number 2 is a root of the characteristic equation (multiplicity 1), so the form is multiplied by x : $y_p = A x e^{2x}$
- The derivatives:
 $y_p' = 2A x e^{2x} + A e^{2x}, y_p'' = 4A x e^{2x} + 4A e^{2x}$
- Substituted into the equation (divided by the

exponential), like terms collected: $A = 3$

- A particular solution: $y_p = 3xe^{2x}$
- The general solution:
 $y = y_h + y_p = C_1 e^x + C_2 e^{2x} + 3xe^{2x}$

Graph



— solutions for different C

The grey curves are solutions for various values of the constants.

The right side $\tan x$ isn't special – variation of constants and the Wronskian.

Input: $y'' + y = \tan x$

Result

$$y = C_1 \cos x + C_2 \sin x - \cos x \cdot \ln |\sec x|$$

Steps

- A linear equation with constant coefficients:
 $y'' + y = \tan x$
- First the homogeneous equation (zero right side):
 $y'' + y = 0$
- We look for solutions $y = e^{\lambda x}$; substituting and dividing by $e^{\lambda x}$ gives the characteristic equation: $\lambda^2 + 1 = 0$
- The roots of the characteristic equation: $\lambda_{1,2} = \pm i$
- Each root λ gives the solution $e^{\lambda x}$, a root of multiplicity k also $x \cdot e^{\lambda x}$, ..., $x^{k-1} \cdot e^{\lambda x}$, and a pair $\alpha \pm \beta i$ the solutions $e^{\alpha x} \cdot \cos \beta x$ and $e^{\alpha x} \cdot \sin \beta x$. The fundamental system: $y_1 = \cos x$, $y_2 = \sin x$
- The homogeneous equation's solution:
 $y_h = C_1 \cos x + C_2 \sin x$
- Variation of constants – we look for the solution as $y = C_1(x) \cos x + C_2(x) \sin x$, the derivatives of the functions C_k satisfy the system

$$\begin{cases} C_1'(x) \cos x + C_2'(x) \sin x = 0 \\ -C_1'(x) \sin x + C_2'(x) \cos x = \tan x \end{cases}$$
- The system's determinant (the Wronskian)

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

- Hence $C_1'(x) = \frac{W_1}{W} = -\sin x \cdot \tan x$
- Hence $C_2'(x) = \frac{W_2}{W} = \sin x$
- The constant taken out of the integral:
 $\int -\sin x \cdot \tan x \, dx = -\int \sin x \cdot \tan x \, dx$
- The function is odd in cosine – the substitution $t = \sin x$
 $dt = \cos x \, dx$
 $\int \sin x \cdot \tan x \, dx = \int \frac{t^2}{(1-t)(t+1)} \, dt$
- The rational function split into a polynomial and partial fractions
 $\int \frac{t^2}{(1-t)(t+1)} \, dt = \int \left(-1 - \frac{1}{2(t-1)} + \frac{1}{2(t+1)} \right) \, dt$
- The integral of a sum is the sum of the integrals
 $\int \left(-1 - \frac{1}{2(t-1)} + \frac{1}{2(t+1)} \right) \, dt = \int -1 \, dt + \int -\frac{1}{2(t-1)} \, dt + \int \frac{1}{2(t+1)} \, dt$
- From the table of basic integrals: $\int -1 \, dt = -t$
- The constant taken out of the integral:
 $\int -\frac{1}{2(t-1)} \, dt = -\frac{1}{2} \int \frac{1}{t-1} \, dt$

- From the table of basic integrals:

$$\int \frac{1}{t-1} dt = \ln |t-1|$$

- The constant taken out of the integral:

$$\int \frac{1}{2(t+1)} dt = \frac{1}{2} \int \frac{1}{t+1} dt$$

- From the table of basic integrals:

$$\int \frac{1}{t+1} dt = \ln |t+1|$$

- Back to the original variable

$$t = \sin x : \quad \frac{\ln |\sin x + 1|}{2} - \frac{\ln |\sin x - 1|}{2} - \sin x = \ln$$

- Integrated (the integration constants are already in the homogeneous solution):

$$C_1(x) = \int -\sin x \cdot \tan x dx = \sin x - \ln \left| \frac{1}{\cos x} + \tan x \right|$$

- From the table of basic integrals: $\int \sin x dx = -\cos x$

- Integrated (the integration constants are already in the homogeneous solution): $C_2(x) = \int \sin x dx = -\cos x$

- A particular solution: $y_p = -\cos x \cdot \ln \left| \frac{1}{\cos x} + \tan x \right|$

- The general solution

$$y = y_h + y_p = C_1 \cos x + C_2 \sin x - \cos x \cdot \ln \left| \frac{1}{\cos x} + \right|$$

An initial value problem – a system for the constants C_1, C_2 .

Input: $y'' - 4y' + 3y = e^{2x}$, $y(0) = 1$, $y'(0) = 0$

Result

$$y = 2e^x - e^{2x}$$

Steps

- A linear equation with constant coefficients:
 $y'' - 4y' + 3y = e^{2x}$
- First the homogeneous equation (zero right side):
 $y'' - 4y' + 3y = 0$
- We look for solutions $y = e^{(\lambda x)}$; substituting and dividing by $e^{(\lambda x)}$ gives the characteristic equation:
 $\lambda^2 - 4\lambda + 3 = 0$
- The discriminant $D = (-4)^2 - 4 \cdot 3 = 4$; the roots:
 $\lambda_1 = 1$, $\lambda_2 = 3$
- Each root λ gives the solution $e^{(\lambda x)}$, a root of multiplicity k also $x \cdot e^{(\lambda x)}$, ..., $x^{(k-1)} \cdot e^{(\lambda x)}$, and a pair $\alpha \pm \beta i$ the solutions $e^{(\alpha x)} \cdot \cos \beta x$ and $e^{(\alpha x)} \cdot \sin \beta x$. The fundamental system: $y_1 = e^x$, $y_2 = e^{3x}$
- The homogeneous equation's solution:
 $y_h = C_1 e^x + C_2 e^{3x}$
- The right side e^{2x} has a special form; the number 2 is not a root of the characteristic equation, so we look for a particular solution $y_p = A e^{2x}$
- The derivatives: $y'_p = 2A e^{2x}$, $y''_p = 4A e^{2x}$
- Substituted into the equation (divided by the exponential), like terms collected: $-A = 1$

• Hence $A = -1$

• A particular solution: $y_p = -e^{2x}$

• The general solution:

$$y = y_h + y_p = C_1 e^x + C_2 e^{3x} - e^{2x}$$

• The general solution's derivatives:

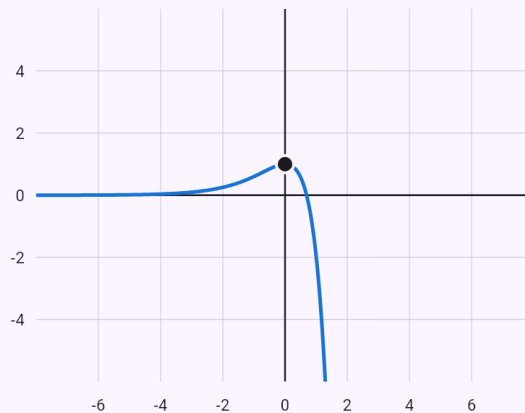
$y' = C_1 e^x - 2e^{2x} + 3C_2 e^{3x}$. The conditions give the system hence $C_1 = 2$, $C_2 = 0$

$$\begin{cases} -1 + C_1 + C_2 = 1 \\ -2 + C_1 + 3C_2 = 0 \end{cases}$$

• The solution satisfying the given conditions:

$$y = 2e^x - e^{2x}$$

Graph



$$y = 2e^x - e^{2x}$$

The highlighted one satisfies the conditions, the points are the given values.

The same problem by the Laplace transform.

Input: $y'' - 4y' + 3y = e^{2x}$, $y(0) = 1$, $y'(0) = 0$

Laplace transform

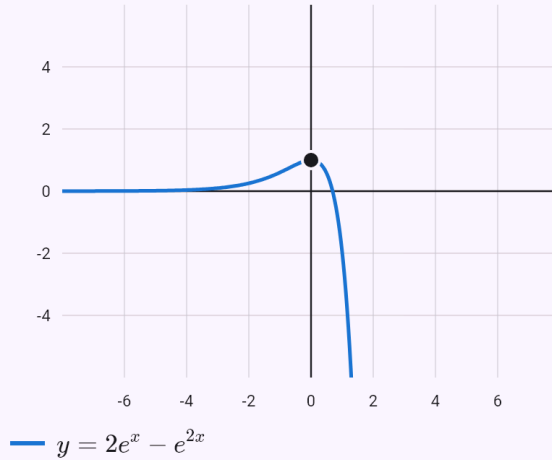
Result

$$y = 2e^x - e^{2x}$$

Steps

- A linear equation with constant coefficients: $y'' - 4y' + 3y = e^{2x}$
- The Laplace transform of both sides: $\mathcal{L}(y) = Y$, $\mathcal{L}(y') = sY - y(0)$, $\mathcal{L}(y'') = s^2Y - sy(0) - y'(0)$; with the initial conditions: $(s^2Y - s) - 4(sY - 1) + 3Y = \frac{1}{s - 2}$
- We express the image Y : $Y = \frac{s - 3}{(s - 2)(s - 1)}$
- Partial fractions: $Y = -\frac{1}{s - 2} + \frac{2}{s - 1}$
- The inverse transform (by the table of images) gives the solution: $y = \mathcal{L}^{-1}\{Y\} = 2e^x - e^{2x}$

Graph



The highlighted one satisfies the conditions, the points are the given values.

A boundary value problem with infinitely many solutions.

Input: $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 0$

Result

$$y = C_2 \sin x, \quad C_2 \in \mathbb{R}$$

Steps

- A linear equation with constant coefficients: $y'' + y = 0$
- We look for solutions $y = e^{\lambda x}$; substituting and dividing by $e^{\lambda x}$ gives the characteristic equation: $\lambda^2 + 1 = 0$
- The roots of the characteristic equation: $\lambda_{1,2} = \pm i$
- Each root λ gives the solution $e^{\lambda x}$, a root of multiplicity k also $x \cdot e^{\lambda x}$, ..., $x^{k-1} \cdot e^{\lambda x}$, and a pair $\alpha \pm \beta i$ the solutions $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$. The fundamental system: $y_1 = \cos x$, $y_2 = \sin x$
- The general solution: $y = C_1 \cos x + C_2 \sin x$
- We substitute the conditions into the general solution hence $C_1 = 0$

$$\begin{cases} C_1 = 0 \\ -C_1 = 0 \end{cases}$$
- The constant C_2 stays arbitrary – the problem has infinitely many solutions.
- The solution satisfying the given conditions: $y = C_2 \sin x$

Euler's equation – the substitution $x = e^t$.

Input: $x^2 y'' - 2xy' + 2y = x^2 + 2$

Result

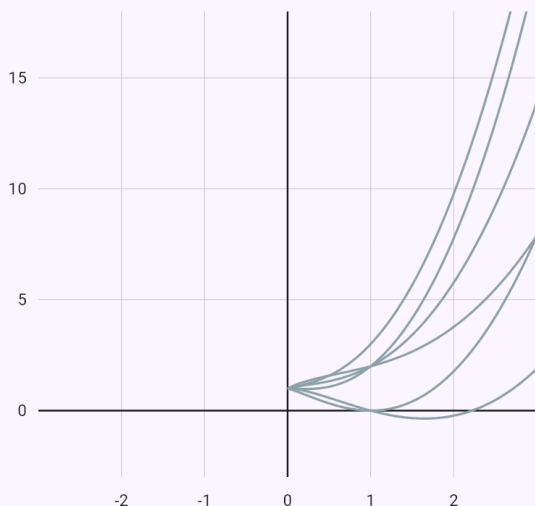
$$y = C_1 x + C_2 x^2 + x^2 \ln x + 1, \quad C_1, C_2$$

Steps

- Euler's equation – each derivative $y^{(k)}$ comes with x^k ; the substitution $x = e^t$ turns it into one with constant coefficients: $x^2 y'' - 2xy' + 2y = x^2 + 2$
- The substitution $x = e^t$, $xy' = \dot{y}$, $x^2 y'' = \ddot{y} - \dot{y}$ turns the equation into $\ddot{y} - 3\dot{y} + 2y = e^{2t} + 2$
- First the homogeneous equation (zero right side): $\ddot{y} - 3\dot{y} + 2y = 0$
- We look for solutions $y = e^{\lambda t}$; substituting and dividing by $e^{\lambda t}$ gives the characteristic equation: $\lambda^2 - 3\lambda + 2 = 0$
- The discriminant $D = (-3)^2 - 4 \cdot 2 = 1$; the roots: $\lambda_1 = 1$, $\lambda_2 = 2$
- Each root λ gives the solution $e^{\lambda t}$, a root of multiplicity k also $x \cdot e^{\lambda t}$, ..., $x^{k-1} \cdot e^{\lambda t}$, and a pair $\alpha \pm \beta i$ the solutions $e^{\alpha t} \cos \beta t$ and $e^{\alpha t} \sin \beta t$. The fundamental system: $y_1 = e^t$, $y_2 = e^{2t}$
- The homogeneous equation's solution: $y_h = C_1 e^t + C_2 e^{2t}$
- We split the right side into parts of a special form – a particular solution for each, then we add them up: $f_1 = 2$, $f_2 = e^{2t}$

- The right side 2 has a special form; the number 0 is not a root of the characteristic equation, so we look for a particular solution $y_{p1} = A$
- The derivatives: $y'_{p1} = 0$, $y''_{p1} = 0$
- Substituted into the equation (divided by the exponential), like terms collected: $2A = 2$
- Hence $A = 1$
- A particular solution: $y_{p1} = 1$
- The right side e^{2t} has a special form; the number 2 is a root of the characteristic equation (multiplicity 1), so the form is multiplied by x : $y_{p2} = Ate^{2t}$
- The derivatives: $y'_{p2} = 2Ate^{2t} + Ae^{2t}$, $y''_{p2} = 4Ate^{2t} + 4Ae^{2t}$
- Substituted into the equation (divided by the exponential), like terms collected: $A = 1$
- A particular solution: $y_{p2} = te^{2t}$
- A particular solution: $y_p = y_{p1} + y_{p2} = te^{2t} + 1$
- The general solution: $y = y_h + y_p = C_1 e^t + C_2 e^{2t} + te^{2t} + 1$
- Back to the original variable ($t = \ln x$, for $x > 0$): $y = C_1 x + C_2 x^2 + x^2 \ln x + 1$

Graph



— solutions for different C

The grey curves are solutions for various values of the constants.

The reduction $y' = p(y)$, the constants from the initial conditions step by step.

Input: $y'' = e^{2y}$, $y(0) = 0$, $y'(0) = 1$

Result

$$y = -\ln(1 - x)$$

Steps

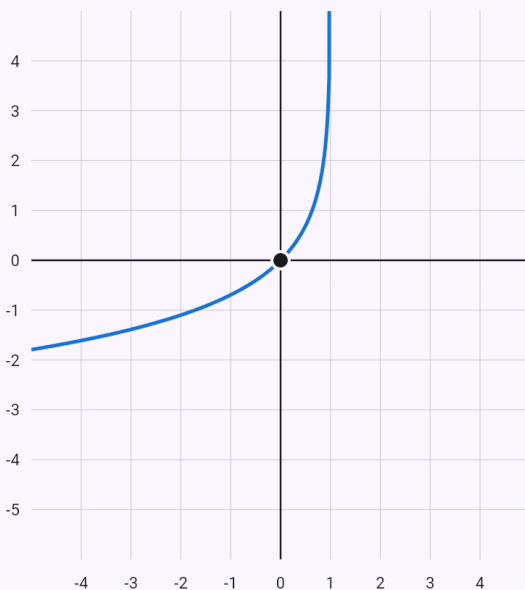
- The independent variable is missing – we take y' as a function of y : $y'' = e^{2y}$
- The substitution $y' = p(y)$, $y'' = p \frac{dp}{dy}$ turns the equation into $p \frac{dp}{dy} = e^{2y}$; the initial conditions give $p(0) = 1$
- A separable equation – the right side is a product of a function of the variable and a function of the unknown: $\frac{dp}{dy} = \frac{e^{2y}}{p}$
- Separating the variables: $p dp = e^{2y} dy$
- Integrating both sides: $\int p dp = \int e^{2y} dy$
- From the table of basic integrals: $\int p dp = \frac{p^2}{2}$
- From the table – the inner function is linear ($ax + b$), so the result is divided by a : $\int e^{2y} dy = \frac{e^{2y}}{2}$
- The general solution in implicit form: $\frac{p^2}{2} = \frac{e^{2y}}{2} + C_1$
- Expressing the solution: $p^2 = e^{2y} + C_1$

- Substituting the initial condition $p(0) = 1$: $1 = C_1 + 1 \Rightarrow C_1 = 0$
- The solution of the initial condition (the particular solution): $p = e^y$
- Back to the original unknown: $y' = e^y$
- A separable equation – the right side is a product of a function of the variable and a function of the unknown: $y' = e^y$
- Separating the variables (dividing, assuming $e^y \neq 0$): $\frac{dy}{e^y} = dx$
- Integrating both sides: $\int \frac{dy}{e^y} = \int dx$
- From the table – the inner function is linear ($ax + b$), so the result is divided by a : $\int e^{-y} dy = -e^{-y}$
- From the table of basic integrals: $\int 1 dx = x$
- The general solution in implicit form: $-e^{-y} = x + C_2$
- Expressing the solution: $y = -\ln(C_2 - x)$
- Substituting the initial condition $y(0) = 0$: $0 = -\ln C_2 \Rightarrow C_2 = 1$
- The solution of the initial condition (the particular solution): $y = -\ln(1 - x)$

Graph

Solutions

Phase plane



— $y = -\ln(1 - x)$

The highlighted one satisfies the conditions, the points are the given values.

A damped oscillator – three cases by δ and ω .

Input: $y'' + 2\delta y' + \omega^2 y = 0$

Result

$$\delta > \omega : y = C_1 e^{(-\delta - \sqrt{\delta^2 - \omega^2})x} + C_2 e^{(-\delta + \sqrt{\delta^2 - \omega^2})x}$$

$$\delta = \omega : y = (C_1 + C_2 x) e^{-\delta x}$$

$$\delta < \omega : y = e^{-\delta x} \left(C_1 \cos \left(\sqrt{\omega^2 - \delta^2} x \right) + C_2 \sin \left(\sqrt{\omega^2 - \delta^2} x \right) \right)$$

Steps

- A linear equation with constant coefficients:

$$y'' + 2\delta y' + \omega^2 y = 0$$

- We look for solutions $y = e^{\lambda x}$; substituting and dividing by $e^{\lambda x}$ gives the characteristic equation:

$$\lambda^2 + 2\delta\lambda + \omega^2 = 0$$

- The roots of the characteristic equation:

$$\lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega^2}$$

- For $\delta > \omega$ so

$$\lambda_1 = -\delta - \sqrt{\delta^2 - \omega^2}, \lambda_2 = -\delta + \sqrt{\delta^2 - \omega^2}$$

$$y = C_1 e^{(-\delta - \sqrt{\delta^2 - \omega^2})x} + C_2 e^{(-\delta + \sqrt{\delta^2 - \omega^2})x}$$

- For $\delta = \omega$: $\lambda_{1,2} = -\delta$, so $y = (C_1 + C_2 x) e^{-\delta x}$

- For $\delta < \omega$: $\lambda_{1,2} = -\delta \pm \sqrt{\omega^2 - \delta^2}i$, so

$$y = e^{-\delta x} \left(C_1 \cos \left(\sqrt{\omega^2 - \delta^2} x \right) + C_2 \sin \left(\sqrt{\omega^2 - \delta^2} x \right) \right)$$

Resonance – for $\omega = 2$ the solution has a different form.

Input: $y'' + \omega^2 y = \cos(2t)$

Result

$$\omega \neq 2 : y = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

$$\omega = 2 : y = C_1 \cos(2t) + C_2 \sin(2t)$$

Steps

- A linear equation with constant coefficients:
 $y'' + \omega^2 y = \cos(2t)$
- First the homogeneous equation (zero right side):
 $y'' + \omega^2 y = 0$
- We look for solutions $y = e^{\lambda t}$; substituting and dividing by $e^{\lambda t}$ gives the characteristic equation: $\lambda^2 + \omega^2 = 0$
- The roots of the characteristic equation: $\lambda_{1,2} = \pm \omega i$
- Each root λ gives the solution $e^{\lambda x}$, a root of multiplicity k also $x \cdot e^{\lambda x}$, ..., $x^{k-1} \cdot e^{\lambda x}$, and a pair $\alpha \pm \beta i$ the solutions $e^{\alpha x} \cdot \cos \beta x$ and $e^{\alpha x} \cdot \sin \beta x$. The fundamental system: $y_1 = \cos(\omega t)$, $y_2 = \sin(\omega t)$
- The homogeneous equation's solution:
 $y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t)$
- The right side $\cos(2t)$ has a special form; the number $2i$ is not a root of the characteristic equation, so we look for a particular solution $y_p = A \cos(2t) + B \sin(2t)$
- The derivatives

$$y'_p = 2B \cos(2t) - 2A \sin(2t), \quad y''_p = -4A \cos(2t) - 4B \sin(2t)$$

- Substituted into the equation (divided by the exponential), like terms collected
 $(A\omega^2 - 4A) \cos(2t) + (B\omega^2 - 4B) \sin(2t) = \cos(2t)$
- We compare the coefficients of like terms on both sides
hence $A = \frac{1}{\omega^2 - 4}$, $B = 0$
$$\begin{cases} A\omega^2 - 4A = 1 \\ B\omega^2 - 4B = 0 \end{cases}$$
- A particular solution: $y_p = \frac{\cos(2t)}{\omega^2 - 4}$
- The general solution
$$y = y_h + y_p = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{\cos(2t)}{\omega^2 - 4}$$
- For $\omega = 2$ the number from the right side is a root of the characteristic equation (resonance) – the solution has a different form: $y = C_1 \cos(2t) + C_2 \sin(2t) + \frac{t \sin(2t)}{4}$

A guessed solution $y_1 = x$ and reduction of order (d'Alembert).

Input: $(1-x^2)y'' - 2xy' + 2y = 0$

Result

$$y = C_1 x + C_2 (x \ln |x-1| - x \ln |x+1| + 2)$$

Steps

- A linear second-order equation with variable coefficients:

$$y'' + \frac{2x}{x^2-1}y' + \frac{2}{1-x^2}y = 0$$

- We guess one solution and check it by substitution:

$y_1 = x$. The second one by reducing the order

(d'Alembert, $y = y_1 \cdot u$)

$$y_2 = y_1 \int \frac{e^{-\int p dx}}{y_1^2} dx = x \int \frac{1}{x^4 - x^2} dx = x \ln |x-1|$$

- The rational function split into a polynomial and partial fractions

$$\int \frac{1}{x^4 - x^2} dx = \int \left(\frac{1}{2(x-1)} - \frac{1}{2(x+1)} - \frac{1}{x^2} \right) dx$$

- The integral of a sum is the sum of the integrals

$$\int \left(\frac{1}{2(x-1)} - \frac{1}{2(x+1)} - \frac{1}{x^2} \right) dx = \int \frac{1}{2(x-1)} dx$$

- The constant taken out of the integral:

$$\int \frac{1}{2(x-1)} dx = \frac{1}{2} \int \frac{1}{x-1} dx$$

- From the table of basic integrals:

$$\int \frac{1}{x-1} dx = \ln |x-1|$$

- The constant taken out of the integral:

$$\int -\frac{1}{2(x+1)} dx = -\frac{1}{2} \int \frac{1}{x+1} dx$$

- From the table of basic integrals:

$$\int \frac{1}{x+1} dx = \ln |x+1|$$

- The constant taken out of the integral:

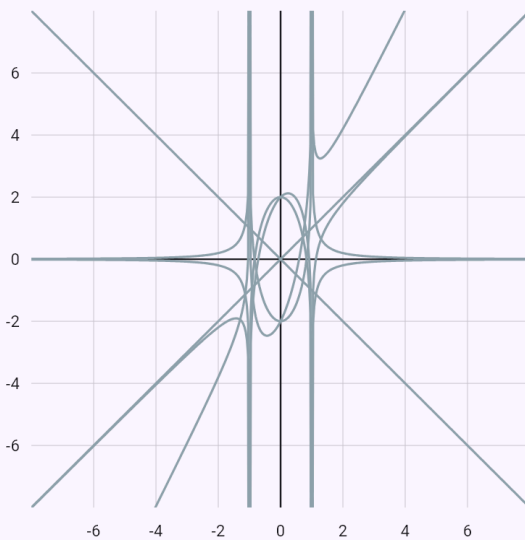
$$\int -\frac{1}{x^2} dx = -\int \frac{1}{x^2} dx$$

- From the table of basic integrals: $\int \frac{1}{x^2} dx = -\frac{1}{x}$

- The general solution:

$$y = C_1 x + C_2 (x \ln |x-1| - x \ln |x+1| + 2)$$

Graph



— solutions for different C

The grey curves are solutions for various values of the constants.

Known limitations

- An equation that needs a hint from the problem (other than $z = \ln y$) can't be solved yet.
- For an equation with a parameter (k, λ, ω) the app draws no graph and takes the parameters as positive.
- Where the solution can't be expressed explicitly, it stays implicit.
- The Laplace transform doesn't know the Heaviside function, shifts and periodic right sides yet.

Recurrences and generating functions

The app solves linear recurrences with constant coefficients of any order by the characteristic equation – multiple and complex roots too (the complex form of the solution and its conversion to the real one by de Moivre's theorem) – and with a right side of the form $P(n) \cdot q^n$ (also $\cos n\varphi$, $\sin n\varphi$, resonance and a sum of right sides). The initial conditions determine the constants; as a check, the app computes further terms from the recurrence and compares them with the formula.

The “Method” chip offers the generating function too ($A(x) = \sum a_n x^n$, partial fractions, geometric series), matrix powers (diagonalization) or, for the first order, summing. A first-order recurrence with a variable coefficient ($a_{n+1} = (n+1)a_n$) is solved by a product (factorials), convertible nonlinear recurrences by the substitution $b_n = 1/a_n$ or $b_n = \ln a_n$. A system of two recurrences is solved by eliminating one sequence or by the matrix, a divide-and-conquer recurrence $T(n) = aT(n/b) + f(n)$ by $n = b^k$ – with the master theorem's estimate.

With the “Generating function” operation the app finds the generating function of a sequence given by a formula ($n^2 + 2^n$) or by its first terms (1, 0, 1, 0, ...), and conversely the formula for a_n from a function $A(x)$. The number of non-negative whole solutions of $x_1 + x_2 + \dots = N$ (also with bounds or coefficients – paying with coins) is the coefficient of a generating function.

The graph shows the terms of the sequence as points (both sequences of a system).

How to enter it

$a(n+2) = 5a(n+1) - 6a(n), a(0) = 1, a(1) = 4$	Brackets, the conditions after a comma.
$a_{\{n+2\}} = 5a_{\{n+1\}} - 6a_n$	Subscripts (also from a photo). Without conditions: the general solution with constants C_1, C_2 .
$F_n = F_{\{n-1\}} + F_{\{n-2\}}, F_1 = F_2 = 1$	The index may go down (a_{n-1}, a_{n-2}); one value can be given to several terms.
$a_{\{n+1\}} = a_n + 2b_n$ $b_{\{n+1\}} = 3a_n + b_n$ $a_0 = 1, b_0 = 0$	A system – the equations on lines, the conditions on the last one.
$T(n) = 2T(n/2) + n, T(1) = 1$	Divide and conquer – a term at n/b .

$n^2 + 2^n$	A sequence's formula with the “Generating function” operation (also $a_n = \dots$).
1, 0, 1, 0, ...	The first terms separated by commas.
$1/(1 - x - x^2)$	A generating function in x (also $A(x) = \dots$) – the formula for a_n .
number of solutions $x_1 + x_2 + x_3 = 10$, $x_1 \leq 3$	The number of non-negative whole solutions with bounds.

Examples

Two distinct real roots, the check by further terms.

Input: $a_{n+2} = 5a_{n+1} - 6a_n$, $a_0 = 1$, $a_1 = 4$

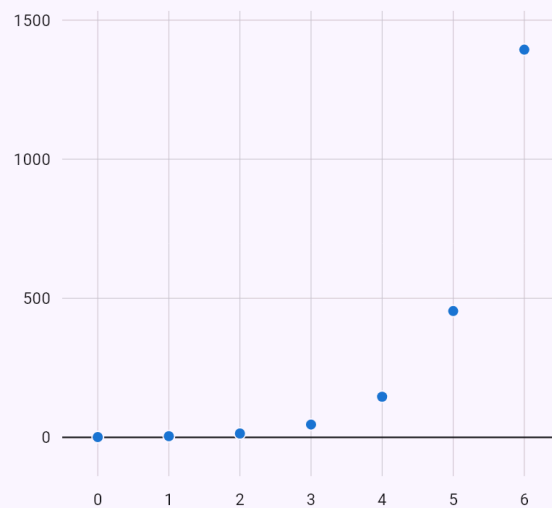
Result

$$a_n = 2 \cdot 3^n - 2^n$$

Steps

- A linear recurrence of order 2 with constant coefficients:
 $a_{n+2} = 5a_{n+1} - 6a_n$
- We look for solutions $a_n = \lambda^n$; substituted and divided by λ^n , they give the characteristic equation:
 $\lambda^2 - 5\lambda + 6 = 0$
- The discriminant $D = (-5)^2 - 4 \cdot 6 = 1$; the roots:
 $\lambda_1 = 2$, $\lambda_2 = 3$
- The general solution: $a_n = C_1 \cdot 2^n + C_2 \cdot 3^n$
- We substitute the conditions into the general solution hence $C_1 = -1$, $C_2 = 2$
 $C_1 + C_2 = 1$
 $2C_1 + 3C_2 = 4$
- The solution satisfying the given conditions:
 $a_n = 2 \cdot 3^n - 2^n$
- Check: the recurrence gives
 $a_2 = 14$, $a_3 = 46$, $a_4 = 146$ – the same as the formula.

Graph



• a_n

The points are the terms computed from the formula – the recurrence itself gives the same values.

The Fibonacci sequence – Binet's formula.

Input: $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$

Result

$$F_n = \frac{\sqrt{5}}{5} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{\sqrt{5}}{5} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

Steps

- A linear recurrence of order 2 with constant coefficients:
 $F_n = F_{n-1} + F_{n-2}$

- We look for solutions $F_n = \lambda^n$; substituted and divided by λ^{n-2} , they give the characteristic equation:
 $\lambda^2 - \lambda - 1 = 0$

- The discriminant $D = (-1)^2 - 4 \cdot (-1) = 5$; the roots:
 $\lambda_1 = \frac{1 - \sqrt{5}}{2}$, $\lambda_2 = \frac{1 + \sqrt{5}}{2}$

- The general solution

$$F_n = C_1 \left(\frac{1 - \sqrt{5}}{2} \right)^n + C_2 \left(\frac{1 + \sqrt{5}}{2} \right)^n$$

- We substitute the conditions into the general solution

$$\text{hence } C_1 = -\frac{\sqrt{5}}{5}, C_2 = \frac{\sqrt{5}}{5}$$
$$C_1 + C_2 = 0$$

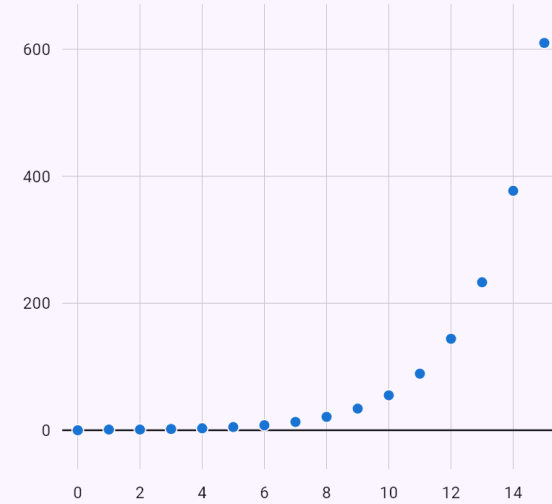
$$\frac{1 - \sqrt{5}}{2} C_1 + \frac{1 + \sqrt{5}}{2} C_2 = 1$$

- The solution satisfying the given conditions

$$F_n = \frac{\sqrt{5}}{5} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{\sqrt{5}}{5} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

- Check: the recurrence gives $F_2 = 1$, $F_3 = 2$, $F_4 = 3$ – the same as the formula.

Graph



● F_n

The points are the terms computed from the formula – the recurrence itself gives the same values.

Complex roots $1 \pm i$ – the complex form and de Moivre's theorem.

Input: $a_{n+2} = 2a_{n+1} - 2a_n$, $a_0 = 1$, $a_1 = 2$

Result

$$a_n = \left(\sqrt{2}\right)^n \left(\cos\left(\frac{\pi n}{4}\right) + \sin\left(\frac{\pi n}{4}\right)\right)$$

Steps

- A linear recurrence of order 2 with constant coefficients:
 $a_{n+2} = 2a_{n+1} - 2a_n$
- We look for solutions $a_n = \lambda^n$; substituted and divided by λ^n , they give the characteristic equation:
 $\lambda^2 - 2\lambda + 2 = 0$
- The discriminant $D = (-2)^2 - 4 \cdot 2 = -4$; the roots:
 $\lambda_{1,2} = 1 \pm i$
- We write the complex roots $1 \pm i$ in polar form: $|\lambda| = \sqrt{2}$, $\varphi = \frac{\pi}{4}$, so $\lambda = \sqrt{2} \left(\cos \frac{\pi}{4} \pm i \sin \frac{\pi}{4}\right)$. By de Moivre's theorem $\lambda^n =$
 $\left(\sqrt{2}\right)^n \left(\cos\left(\frac{\pi n}{4}\right) \pm i \sin\left(\frac{\pi n}{4}\right)\right)$
- The general solution in the complex form (complex constants): $a_n = C_1(1+i)^n + C_2(1-i)^n$. Adding the conjugate terms, with new real constants we get the real form
 $a_n = \left(\sqrt{2}\right)^n \left(C_1 \cos\left(\frac{\pi n}{4}\right) + C_2 \sin\left(\frac{\pi n}{4}\right)\right)$
- We substitute the conditions into the general solution hence $C_1 = 1$, $C_2 = 1$

The points are the terms computed from the formula – the recurrence itself gives the same values.

$$C_1 = 1$$

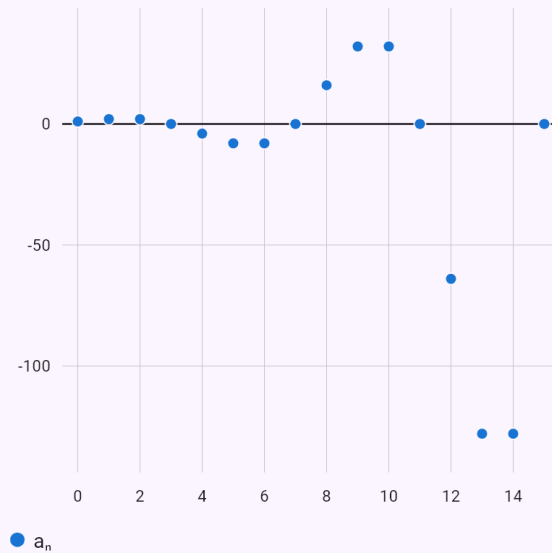
$$C_1 + C_2 = 2$$

- The solution satisfying the given conditions

$$a_n = \left(\sqrt{2}\right)^n \left(\cos\left(\frac{\pi n}{4}\right) + \sin\left(\frac{\pi n}{4}\right)\right)$$

- Check: the recurrence gives $a_2 = 2$, $a_3 = 0$, $a_4 = -4$ – the same as the formula.

Graph



The right side n, a double root 2.

Input: $a_{n+2} = 4a_{n+1} - 4a_n + n$, $a_0 = 0$, $a_1 = 0$

Result

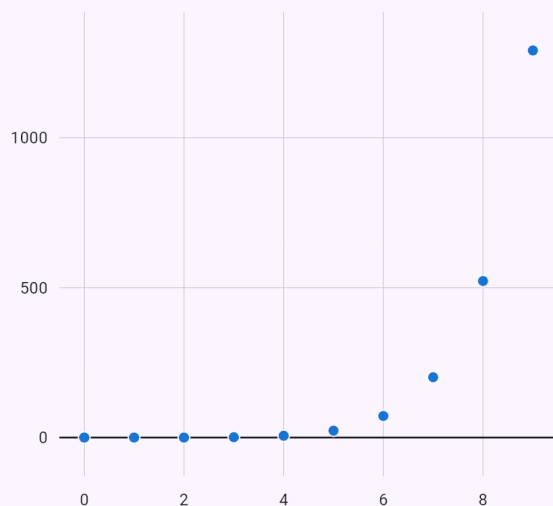
$$a_n = \frac{n-4}{2} \cdot 2^n + n + 2$$

Steps

- A linear recurrence of order 2 with constant coefficients:
 $a_{n+2} = 4a_{n+1} - 4a_n + n$
- First we solve the homogeneous recurrence (the right side 0): $a_{n+2} - 4a_{n+1} + 4a_n = 0$
- We look for solutions $a_n = \lambda^n$; substituted and divided by λ^n , they give the characteristic equation:
 $\lambda^2 - 4\lambda + 4 = 0$
- The discriminant $D = (-4)^2 - 4 \cdot 4 = 0$; the roots:
 $\lambda_{1,2} = 2$
- The homogeneous recurrence's general solution:
 $a_n^{(h)} = (C_1 + C_2 n) \cdot 2^n$
- The right side n has a special form; the number 1 is not a root of the characteristic equation, so we look for a particular solution $a_n^{(p)} = An + B$
- Substituted into the recurrence, the terms collected:
 $An + B - 2A = n$
- We compare the coefficients of like terms on both sides
hence $A = 1$, $B = 2$
 $A = 1$
 $B - 2A = 0$

- A particular solution: $a_n^{(p)} = n + 2$
- The general solution:
 $a_n = a_n^{(h)} + a_n^{(p)} = (C_1 + C_2 n) \cdot 2^n + n + 2$
- We substitute the conditions into the general solution
hence $C_1 = -2$, $C_2 = \frac{1}{2}$
 $C_1 + 2 = 0$
 $2C_1 + 2C_2 + 3 = 0$
- The solution satisfying the given conditions:
 $a_n = \frac{n-4}{2} \cdot 2^n + n + 2$
- Check: the recurrence gives $a_2 = 0$, $a_3 = 1$, $a_4 = 6$ – the same as the formula.

Graph



• a_n

The points are the terms computed from the formula – the recurrence itself gives the same values.

The same recurrence by the generating function.

Input: $a_{n+2} = 5a_{n+1} - 6a_n$, $a_0 = 1$, $a_1 = 4$

Generating function

Result

$$a_n = 2 \cdot 3^n - 2^n$$

Steps

- A linear recurrence of order 2 with constant coefficients:

$$a_{n+2} = 5a_{n+1} - 6a_n$$

- We introduce the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$

- The recurrence multiplied by x^{n+2} and summed over all $n \geq 0$

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} = 5 \sum_{n=0}^{\infty} a_{n+1} x^{n+2} - 6 \sum_{n=0}^{\infty} a_n x^{n+2}$$

- The sums expressed by the generating function and the initial values:

$$A(x) - 1 - 4x = 5x(A(x) - 1) - 6x^2 A(x)$$

- The generating function expressed, its denominator factored: $A(x) = \frac{1-x}{(1-2x)(1-3x)}$

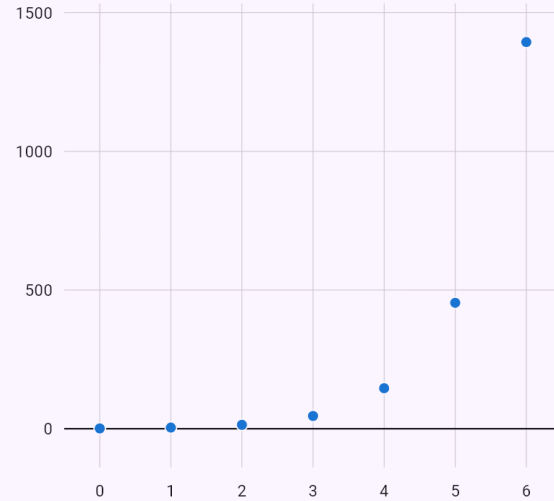
- Partial fractions: $A(x) = -\frac{1}{1-2x} + \frac{2}{1-3x}$

- The fractions as geometric series

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n, \quad \frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n$$

- The coefficient of x^n is the term: $a_n = 2 \cdot 3^n - 2^n$

Graph



• a_n

The points are the terms computed from the formula – the recurrence itself gives the same values.

The same recurrence by matrix powers.

Input: $a_{n+2} = 5a_{n+1} - 6a_n$, $a_0 = 1$, $a_1 = 4$

Matrix powers

Result

$$a_n = 2 \cdot 3^n - 2^n$$

Steps

- A linear recurrence of order 2 with constant coefficients:

$$a_{n+2} = 5a_{n+1} - 6a_n$$

- The recurrence in matrix form i.e. $v_{n+1} = M \cdot v_n$ with

$$\begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$$

$$M = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}$$

- Substituting again and again, $v_n = M^n \cdot v_0$

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = M^n \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

- The matrix's eigenvalues:

$$\det(M - \lambda E) = \lambda^2 - 5\lambda + 6 = 0, \text{ so } \lambda_1 = 2, \lambda_2 = 3$$

- The eigenvectors – the solutions of $(M - \lambda E)u = 0$

$$u_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

- The matrix diagonalized, $M = P \cdot D \cdot P^{-1}$

$$P = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$$

- The matrix power the coefficients so $v_n = P \cdot D^n \cdot (P^{-1}v_0)$.

$$M^n = P D^n P^{-1}, \quad D^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

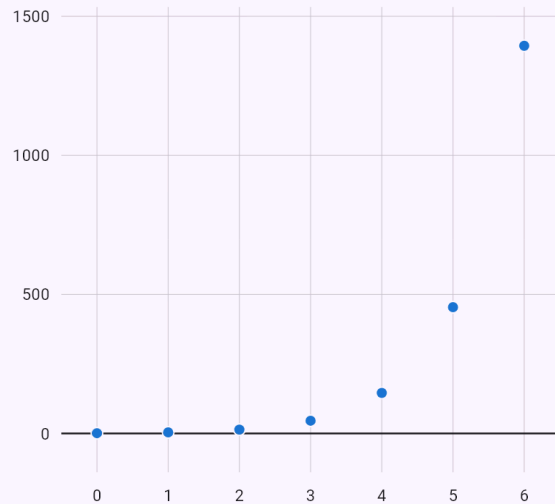
- The solution satisfying the given conditions:

$$a_n = 2 \cdot 3^n - 2^n$$

- Check: the recurrence gives

$$a_2 = 14, \quad a_3 = 46, \quad a_4 = 146 - \text{the same as the formula.}$$

Graph



• a_n

The points are the terms computed from the formula – the recurrence itself gives the same values.

A first-order recurrence by summing.

Input: $a_{n+1} = 2a_n + 1$, $a_0 = 0$

Summing

Result

$$a_n = 2^n - 1$$

Steps

- A linear recurrence of order 1 with constant coefficients:

$$a_{n+1} = 2a_n + 1$$

- Substituting the recurrence into itself again and again:

$$a_n = 2a_{n-1} + 1 = 4a_{n-2} + \dots, \text{ in general}$$

$$a_n = a_0 \cdot 2^n + \sum_{j=0}^{n-1} 2^{n-1-j}$$

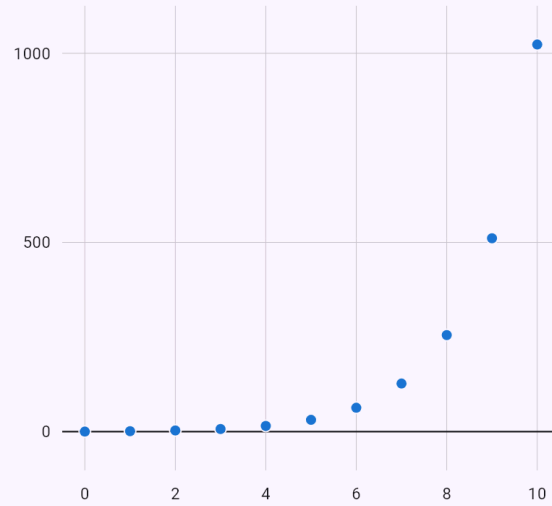
- The sum evaluated: $\sum_{j=0}^{n-1} 2^{n-1-j} = 2^n - 1$

- The solution satisfying the given conditions:

$$a_n = 2^n - 1$$

- Check: the recurrence gives $a_1 = 1$, $a_2 = 3$, $a_3 = 7$ – the same as the formula.

Graph



• a_n

The points are the terms computed from the formula – the recurrence itself gives the same values.

A variable coefficient – the product gives a factorial.

Input: $a_{n+1} = (n+1)a_n$, $a_0 = 1$

Result

$$a_n = n!$$

Steps

- A first-order linear recurrence with a variable coefficient:

$$a_{n+1} = a_n(n+1)$$

- Substituting the recurrence into itself again and again – the term is a product

$$a_n = (n+1)a_{n-1} = n(n+1)a_{n-2} = \dots = a_0 \prod_{j=0}^{n-1} (j+1)$$

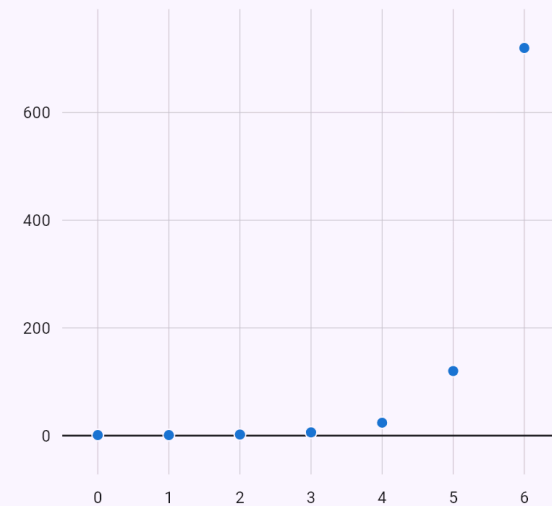
- The product simplified (the factorials cancel):

$$\prod_{j=0}^{n-1} (j+1) = n!$$

- The solution satisfying the given conditions: $a_n = n!$

- Check: the recurrence gives $a_1 = 1$, $a_2 = 2$, $a_3 = 6$ – the same as the formula.

Graph



• a_n

The points are the terms computed from the formula – the recurrence itself gives the same values.

A nonlinear recurrence – the substitution $b_n = 1/a_n$.

Input: $a_{n+1} = a_n/(1 + a_n)$, $a_0 = 1$

Result

$$a_n = \frac{1}{n+1}$$

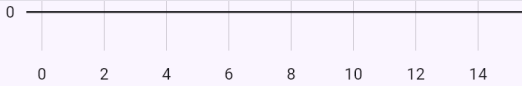
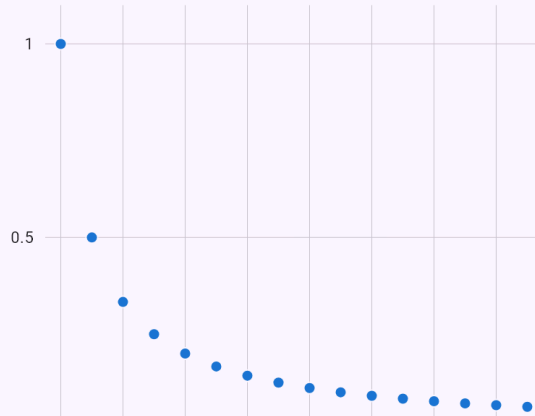
Steps

- A nonlinear recurrence – the reciprocals of the terms make it linear: $a_{n+1} = \frac{a_n}{a_n + 1}$
- Both sides reciprocated: $\frac{1}{a_{n+1}} = \frac{1}{a_n} + 1$; the substitution $b_n = \frac{1}{a_n}$ thus makes the recurrence linear: $b_{n+1} = b_n + 1$
- The initial conditions for the new sequence: $b_0 = 1$
- First we solve the homogeneous recurrence (the right side 0): $b_{n+1} - b_n = 0$
- We look for solutions $b_n = \lambda^n$; substituted and divided by λ^n , they give the characteristic equation: $\lambda - 1 = 0$
- The roots of the characteristic equation: $\lambda_1 = 1$
- The homogeneous recurrence's general solution: $b_n^{(h)} = C_1$
- The right side 1 has a special form; the number 1 is a root of the characteristic equation (multiplicity 1), so the form is multiplied by n: $b_n^{(p)} = An$
- Substituted into the recurrence, the terms collected:

$$A = 1$$

- A particular solution: $b_n^{(p)} = n$
- The general solution: $b_n = b_n^{(h)} + b_n^{(p)} = C_1 + n$
- We substitute the conditions into the general solution hence $C_1 = 1$
 $C_1 = 1$
- The solution satisfying the given conditions: $b_n = 1 + n$
- Check: the recurrence gives $b_1 = 2$, $b_2 = 3$, $b_3 = 4$ – the same as the formula.
- Back to the original sequence: $a_n = \frac{1}{b_n} = \frac{1}{n+1}$

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

A system of two recurrences by eliminating the sequence b_n .

Input: $a_{n+1} = a_n + 2b_n$

$b_{n+1} = 3a_n + b_n$

$$a_0 = 1, b_0 = 0$$

Result

$$a_n = \frac{(1 - \sqrt{6})^n}{2} + \frac{(1 + \sqrt{6})^n}{2}$$

$$b_n = \frac{\sqrt{6}}{4} (1 + \sqrt{6})^n - \frac{\sqrt{6}}{4} (1 - \sqrt{6})^n$$

Steps

- A system of two linear recurrences

$$a_{n+1} = a_n + 2b_n$$

$$b_{n+1} = 3a_n + b_n$$

- From the first equation $b_n = \frac{a_{n+1} - a_n}{2}$, so

$b_{n+1} = \frac{a_{n+2} - a_{n+1}}{2}$; put into the second one they give a recurrence for a single sequence:

$$a_{n+2} - 2a_{n+1} - 5a_n = 0$$

- The second initial condition from the first equation:

$$a_1 = 1 + 0 = 1$$

- We look for solutions $a_n = \lambda^n$; substituted and divided by λ^n , they give the characteristic equation:

$$\lambda^2 - 2\lambda - 5 = 0$$

- The discriminant $D = (-2)^2 - 4 \cdot (-5) = 24$; the roots:

$$\lambda_1 = 1 - \sqrt{6}, \lambda_2 = 1 + \sqrt{6}$$

- The general solution:

$$a_n = C_1 (1 - \sqrt{6})^n + C_2 (1 + \sqrt{6})^n$$

- We substitute the conditions into the general solution

$$\text{hence } C_1 = \frac{1}{2}, C_2 = \frac{1}{2}$$

$$C_1 + C_2 = 1$$

$$(1 - \sqrt{6})C_1 + (1 + \sqrt{6})C_2 = 1$$

- The solution satisfying the given conditions:

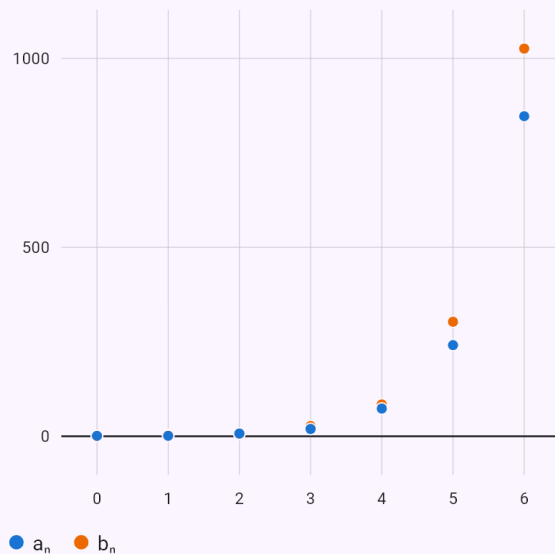
$$a_n = \frac{(1 - \sqrt{6})^n}{2} + \frac{(1 + \sqrt{6})^n}{2}$$

- Check: the recurrence gives $a_2 = 7$, $a_3 = 19$, $a_4 = 73$ – the same as the formula.

- The other sequence from the expression

$$b_n = \frac{a_{n+1} - a_n}{2} = \frac{\sqrt{6}}{4} (1 + \sqrt{6})^n - \frac{\sqrt{6}}{4} (1 - \sqrt{6})^n$$

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

Divide and conquer (merge sort) – $n = 2^k$ and the master theorem.

Input: $T(n) = 2T(n/2) + n$, $T(1) = 1$

Result

$$T(n) = n + n \log_2 n$$

Conditions: $n = 2^k$, $k \in \mathbb{N}_0$

Steps

- A divide-and-conquer recurrence: $T(n) = 2T\left(\frac{n}{2}\right) + n$
- For n a power of the base: the substitution $n = 2^k$, $S_k = T(2^k)$ gives a first-order linear recurrence: $S_{k+1} = 2S_k + 2^{k+1}$
- The initial conditions for the new sequence: $S_0 = 1$
- First we solve the homogeneous recurrence (the right side 0): $S_{k+1} - 2S_k = 0$
- We look for solutions $S_k = \lambda^k$; substituted and divided by λ^k , they give the characteristic equation: $\lambda - 2 = 0$
- The roots of the characteristic equation: $\lambda_1 = 2$
- The homogeneous recurrence's general solution: $S_k^{(h)} = C_1 \cdot 2^k$
- The right side 2^{k+1} has a special form; the number 2 is a root of the characteristic equation (multiplicity 1), so the form is multiplied by k : $S_k^{(p)} = Ak \cdot 2^k$
- Substituted into the recurrence, divided by 2^k , the terms collected: $2A = 2$
- Hence $A = 1$

- A particular solution: $S_k^{(p)} = k \cdot 2^k$
- The general solution:
 $S_k = S_k^{(h)} + S_k^{(p)} = C_1 \cdot 2^k + k \cdot 2^k$
- We substitute the conditions into the general solution hence $C_1 = 1$
 $C_1 = 1$
- The solution satisfying the given conditions:
 $S_k = 2^k + k \cdot 2^k$
- Check: the recurrence gives $S_1 = 4$, $S_2 = 12$, $S_3 = 32$ – the same as the formula.
- Back to n ($k = \log_2 n$): $T(n) = n + n \log_2 n$
- The master theorem ($a = 2$, $b = 2$, $f(n) = \Theta(n)$): $2 = 2^1$, so $T(n) = \Theta(n \log n)$

The generating function of the sequence $n^2 + 2^n$.

Input: $n^2 + 2^n$ Generating function

Result

$$A(x) = \frac{1 - 2x + 2x^2 - 3x^3}{(1-x)^3(1-2x)}$$

Steps

- We introduce the generating function

$$A(x) = \sum_{n=0}^{\infty} (n^2 + 2^n)x^n$$

- The series split by linearity:

$$A(x) = \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} n^2 x^n$$

- A known series: $\sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}$

- A known series: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

- Multiplying the terms by n is differentiating and multiplying by x

$$\sum_{n=0}^{\infty} n x^n = x \cdot \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{x}{(1-x)^2}$$

- Multiplying the terms by n is differentiating and multiplying by x

$$\sum_{n=0}^{\infty} n^2 x^n = x \cdot \frac{d}{dx} \left(\frac{x}{(1-x)^2} \right) = \frac{x+x^2}{(1-x)^3}$$

- The generating function: $A(x) = \frac{1-2x+2x^2-3x^3}{(1-x)^3(1-2x)}$
- Check – the expansion starts with the sequence's terms:
 $A(x) = 1 + 3x + 8x^2 + 17x^3 + \dots$

The formula for a_n from a generating function – partial fractions by irrational roots.

Input: $1/(1 - x - x^2)$ Generating function

Result

$$a_n = \frac{5 - \sqrt{5}}{10} \left(\frac{1 - \sqrt{5}}{2} \right)^n + \frac{5 + \sqrt{5}}{10} \left(\frac{1 + \sqrt{5}}{2} \right)^n$$

Steps

- The generating function expressed, its denominator factored

$$A(x) = \frac{1}{\left(1 + \frac{-1+\sqrt{5}}{2}x\right)\left(1 - \frac{1+\sqrt{5}}{2}x\right)}$$

- Partial fractions

$$A(x) = \frac{\frac{5-\sqrt{5}}{10}}{1 + \frac{-1+\sqrt{5}}{2}x} + \frac{\frac{5+\sqrt{5}}{10}}{1 - \frac{1+\sqrt{5}}{2}x}$$

- The fractions as geometric series

$$\frac{1}{1 + \frac{-1+\sqrt{5}}{2}x} = \sum_{n=0}^{\infty} \left(\frac{1-\sqrt{5}}{2} \right)^n x^n, \quad \frac{1}{1 - \frac{1+\sqrt{5}}{2}x} = \sum_{n=0}^{\infty} \left(\frac{1+\sqrt{5}}{2} \right)^n x^n$$

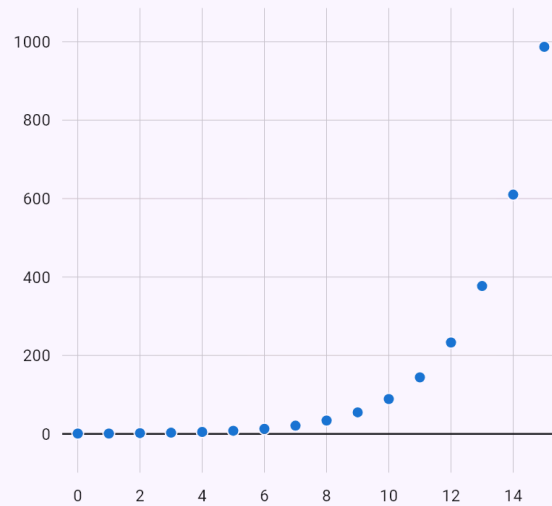
- The coefficient of x^n is the term

$$a_n = \frac{5 - \sqrt{5}}{10} \left(\frac{1 - \sqrt{5}}{2} \right)^n + \frac{5 + \sqrt{5}}{10} \left(\frac{1 + \sqrt{5}}{2} \right)^n$$

- Check – the expansion starts with the sequence's terms:

$$A(x) = 1 + x + 2x^2 + 3x^3 + \dots$$

Graph



• a_n

The points are the terms computed from the formula – the recurrence itself gives the same values.

The number of solutions of $x_1 + x_2 + x_3 = 10$ with $x_1 \leq 3$.

Input: number of solutions $x_1 + x_2 + x_3 = 10$, $x_1 \leq 3$

Result

38

Steps

- The unknown x_1 ($0 \leq x_1 \leq 3$) contributes the series

$$1 + x + x^2 + x^3 = \frac{1 - x^4}{1 - x}$$

- The unknown x_2 ($x_2 \geq 0$) contributes the series

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1 - x}$$

- The unknown x_3 ($x_3 \geq 0$) contributes the series

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1 - x}$$

- The generating function is their product – the coefficient

$$\text{of } x^N \text{ counts the solutions: } G(x) = \frac{1 - x^4}{(1 - x)^3}$$

- The number of solutions is the coefficient of x^{10} : 38

Known limitations

- Not yet: higher-order linear recurrences with variable coefficients, systems of three or more recurrences, divide and conquer with several terms ($T(n/2)$ and $T(n/3)$) and nonlinear recurrences these substitutions don't make linear.

Applications of differential equations

The "Applications of differential equations" button on the home screen opens a form with 23 models: radiocarbon dating, decay and half-life, exponential growth, a decay chain, Newton's cooling (also the time of death from two measurements), chemical kinetics of the 0th–2nd order, a drug in the blood (one dose, repeated doses, an infusion), blood alcohol, savings and loans, RC, RL and RLC circuits, a mass on a spring, a fall with air resistance, a draining tank, mixing in a tank, logistic and Gompertz growth, a harvested population, the SIR epidemic, predator and prey, two competing species and a chemostat.

You choose the model, its variant and what you are looking for; the constants (^{14}C 's half-life, the gravitational acceleration, ...) are prefilled and can be changed. Every quantity and the result have a unit to choose – the app converts the values (the conversion is a step). The "?" by the model explains where its equation comes from.

The steps start by setting the model up (why this equation), go on solving the equation (separation of variables, an integrating factor, the characteristic equation), substitute the conditions, express the sought quantity and end by substituting the numbers. Systems (SIR, predator and prey, competition, a chemostat) are solved numerically by the Runge–Kutta method, their equilibria and stability exactly. The graph shows the quantities in time with the answer marked, for systems also the phase plane.

Examples

The age of a find with 35 % of ^{14}C left.

Input: `model(radiocarbon; find=t[yr]; T12=5730[yr]; p=35[pct]; input=ratio)`

Result

$$t \approx 8700 \text{ yr}$$

Steps

- Find t ; given: $T_{1/2} = 5730 \text{ yr}$, $\frac{N}{N_0} = 35 \%$
- Converting the units: $\frac{N}{N_0} = 35 \% = 0.35$
- Living organisms take in carbon from the air, so they contain the same share of the radioactive isotope ^{14}C as the atmosphere. After death the intake stops and the ^{14}C decays. The number of decays per unit of time is proportional to the number of nuclei left: $dN/dt = -\lambda N$, λ being the decay constant. The solution is $N = N_0 \cdot e^{(-\lambda t)}$, and half of it is gone after the half-life $T_{1/2} = 5730$ years, so $\lambda = \ln 2 / T_{1/2}$. From the share of ^{14}C left (or from the measured activity against a living sample's) we get the age. The model's equation: $\frac{dN}{dt} = -\lambda N$
- Separating the variables: $\frac{dN}{N} = -\lambda dt$
- Integrating both sides: $\ln |N| = -\lambda t + C$
- Exponentiating – the constant $\pm e^C$ is called C again: $N = C e^{-\lambda t}$
- Substituting the initial condition $N(0) = N_0$: $N_0 = C$
- The solution of the initial condition (the particular solution): $N = N_0 e^{-\lambda t}$

- In a half-life the amount halves – the decay constant

$$N(T_{1/2}) = \frac{N_0}{2} \Rightarrow \lambda = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{5730} \approx 1.21 \cdot 10^{-4} \text{ yr}^{-1}$$

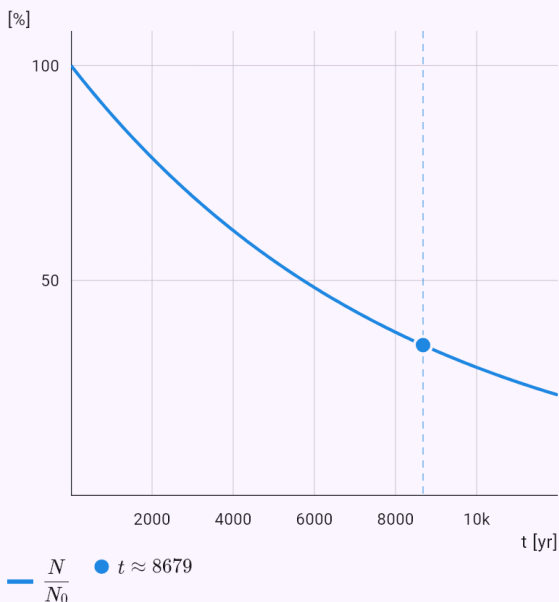
- Expressing the sought quantity:

$$t = -\frac{1}{\lambda} \ln \frac{N}{N_0} = \frac{T_{1/2}}{\ln 2} \ln \frac{N_0}{N}$$

- Substituting: $t = \frac{5730}{\ln 2} \ln \frac{1}{0.35} \approx 8679 \text{ yr}$

- Laboratories give the "radiocarbon age" with Libby's half-life of 5568 years and convert it by a calibration curve (the atmosphere's ^{14}C hasn't always been the same) – the true age may therefore differ somewhat.

Graph



The model's quantities in time, computed from the solution; the answer is marked.

The time of death: the body was 30 °C when found, 28 °C an hour later, the room 20 °C.

Input: `model(cooling; find=t[h]; T0=30[degC]; Ta=20[degC]; t1=1[h]; T1=28[degC]; Tt=37[degC]; k=measured)`

Result

$$t \approx -2.38 \text{ h}$$

Steps

- Find t , given

$$T_0 = 30^\circ\text{C}, \quad T_a = 20^\circ\text{C}, \quad t_1 = 1 \text{ h}, \quad T_1 = 28^\circ\text{C},$$

- A body gives off heat the faster, the bigger the difference of temperatures: $dT/dt = -k(T - T_a)$, T_a being the ambient temperature and k depending on the body (size, material, air flow). The solution is $T = T_a + (T_0 - T_a) \cdot e^{-(kt)}$ – the difference falls exponentially. k can be found from two measurements. That's how forensic scientists estimate the time of death: they measure the body's temperature when found and a bit later and calculate when it was 37 °C. The model's equation: $\frac{dT}{dt} = -k(T - T_a)$

- Separating the variables: $\frac{dT}{T - T_a} = -k dt$

- Integrating both sides: $\ln |T - T_a| = -kt + C$

- Exponentiating – the constant $\pm e^C$ is called C again:

$$T = T_a + C e^{-kt}$$

- Substituting the initial condition $T(0) = T_0$:

$$T_0 = T_a + C \Rightarrow C = T_0 - T_a$$

- The solution of the initial condition (the particular solution): $T = T_a + (T_0 - T_a) e^{-kt}$

- k from the second measurement

$$T(t_1) = T_1 \Rightarrow k = -\frac{1}{t_1} \ln \frac{T_1 - T_a}{T_0 - T_a} = -\frac{1}{1} \ln \frac{28 - 20}{30 - 20} \approx$$

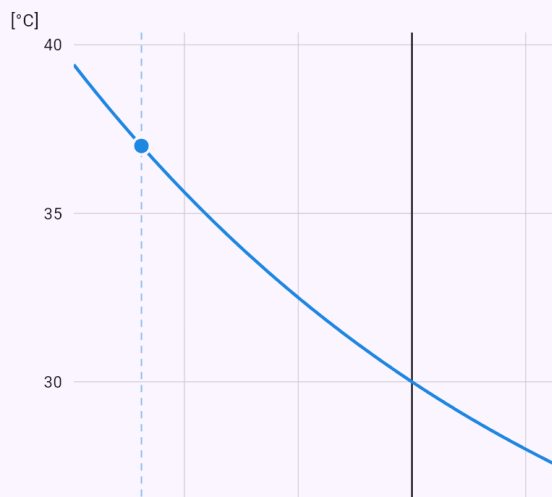
- Expressing the sought quantity

$$T_a + (T_0 - T_a) e^{-kt} = T \Rightarrow t = -\frac{1}{k} \ln \frac{T - T_a}{T_0 - T_a}$$

- Substituting: $t = -\frac{1}{0.2231} \ln \frac{37 - 20}{30 - 20} \approx -2.378 \text{ h}$

- A negative time means before the first measurement (for the time of death: that many hours before the body was found).

Graph



— T • $t \approx -2.378$

The model's quantities in time, computed from the solution; the answer is marked.

A 500 mg drug every 8 hours – the steady level between doses.

Input: `model(drug; find=steady[mg_l]; D=500[mg]; V=40[l]; T12=6[h]; tau=8[h];`

mode=repeated)

Result

$$c_{max} \approx 20.7 \text{ mg/l}$$

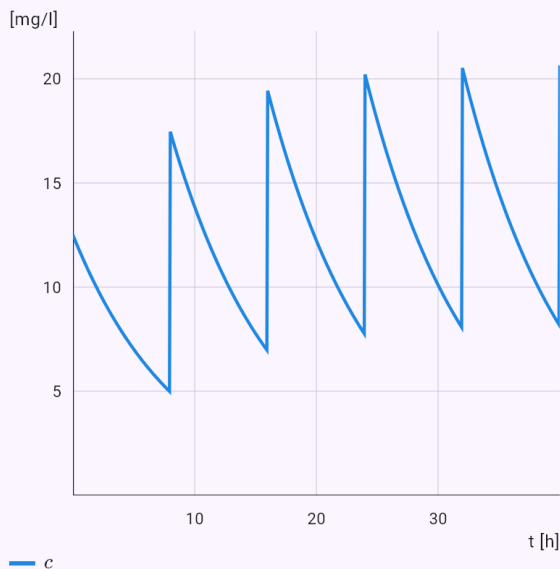
$$c_{min} \approx 8.22 \text{ mg/l}$$

Steps

- Find c_{max} ; given
 $D = 500 \text{ mg}$, $V = 40 \text{ l}$, $T_{1/2} = 6 \text{ h}$, $\tau = 8 \text{ h}$
- A drug in the blood spreads in the volume of distribution V ; the kidneys and the liver eliminate it at a rate proportional to the concentration: $dc/dt = -kc$, $k = \ln 2 / T_{1/2}$ (the biological half-life). After one dose D , $c = (D/V) \cdot e^{-(k\tau)}$. With repeated doses every τ the remainders add up (a geometric series) and the level settles between c_{min} and c_{max} . An infusion at a rate R gives $dc/dt = R/V - kc$, the concentration approaching the steady value $R/(kV)$. The model's equation: $\frac{dc}{dt} = -kc$
- Initial concentration after the dose:
 $c_0 = \frac{D}{V} = \frac{500}{40} \approx 12.5 \text{ mg/l}$
- Elimination constant from the biological half-life:
 $k = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{6} \approx 0.1155 \text{ h}^{-1}$
- Separating the variables: $\frac{dc}{c} = -k dt$
- Integrating both sides: $\ln |c| = -kt + C$

- Exponentiating – the constant $\pm e^C$ is called C again:
 $c = C e^{-kt}$
- Substituting the initial condition $c(0) = c_0$: $c_0 = C$
- The solution of the initial condition (the particular solution): $c = c_0 e^{-kt}$
- Share of the drug left by the next dose:
 $q = \frac{c(\tau)}{c_0} = e^{-k\tau} = e^{-0.1155 \cdot 8} \approx 0.3969$
- Expressing the sought quantity
 $c_{max} = c_0(1 + q + q^2 + \dots) = \frac{c_0}{1 - q}$, $c_{min} = q c_{max}$
- Substituting: $c_{max} = \frac{12.5}{1 - 0.3969} \approx 20.72 \text{ mg/l}$
- Substituting: $c_{min} = 0.3969 \cdot 20.72 \approx 8.225 \text{ mg/l}$
- Accumulation factor (how many times the steady level exceeds one dose's): $R = \frac{1}{1 - q} \approx 1.658$

Graph



The model's quantities in time, computed from the solution; the answer is marked.

The monthly repayment of a loan of 1 000 000 CZK for 20 years at 5 %.

Input: `model(interest; find=a[CZK_mo]; B0=1000000[CZK]; r=5[pct_yr]; t=20[yr]; B=0[CZK];`

kind=loan)

Result

$$a \approx 6592 \text{ CZK/mo}$$

Steps

- Find a ; given

$$D_0 = 1\,000\,000 \text{ CZK}, \quad r = 5\% \text{ p.a.}, \quad t = 20 \text{ yr}, \quad D$$

- Converting the units: $r = 5\% \text{ p.a.} = 0.05 \text{ yr}^{-1}$
- With continuous compounding, interest accrues at every moment in proportion to the balance: $dB/dt = rB$, r being the annual rate. Regular deposits a raise the balance ($dB/dt = rB + a$), a loan's repayments lower the debt ($dB/dt = rB - a$). The equation is solved by separating the variables, and the solution gives the balance, the time to repay or the payment (the continuous counterpart of an annuity). The model's equation: $\frac{dD}{dt} = rD - a$
- Separating the variables: $\frac{dD}{D - \frac{a}{r}} = r dt$
- Integrating both sides: $\ln \left| D - \frac{a}{r} \right| = rt + C$
- Exponentiating – the constant $\pm e^C$ is called C again:
 $D = \frac{a}{r} + Ce^{rt}$
- Substituting the initial condition $D(0) = D_0$:
 $C = D_0 - \frac{a}{r}$

- The solution of the initial condition (the particular solution): $D = \frac{a}{r} + \left(D_0 - \frac{a}{r} \right) e^{rt}$

- Continuous compounding is a model – banks compound monthly or yearly; the results differ only a little.

- Expressing the sought quantity: $a = \frac{r(D_0 e^{rt} - D)}{e^{rt} - 1}$

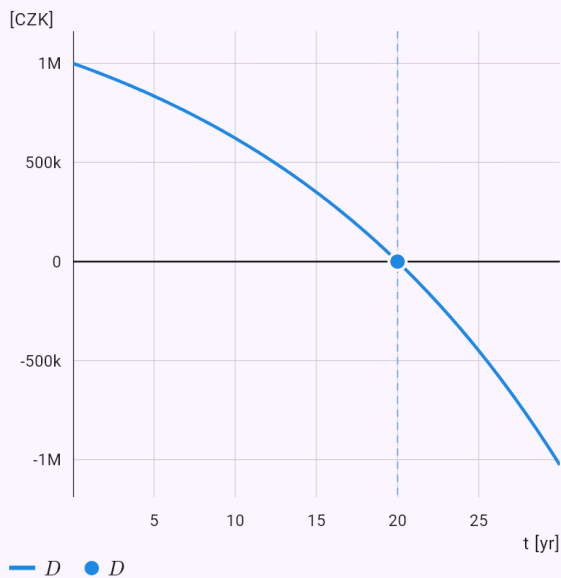
- Substituting

$$a = \frac{0.05(1\,000\,000 e^{0.05 \cdot 20} - 0)}{e^{0.05 \cdot 20} - 1} \approx 79\,100 \text{ CZK/yr}$$

- Paid in total: $a t = 1\,582\,000 \text{ CZK}$

- Converting the units:
 $a \approx 79\,100 \text{ CZK/yr} = 6592 \text{ CZK/mo}$

Graph



The model's quantities in time, computed from the solution; the answer is marked.

How long a capacitor takes to charge to half the source voltage.

Input: `model(rc; find=t[ms]; R=10[kohm]; C=100[uF]; U=12[V]; u=6[V]; mode=charge)`

Result

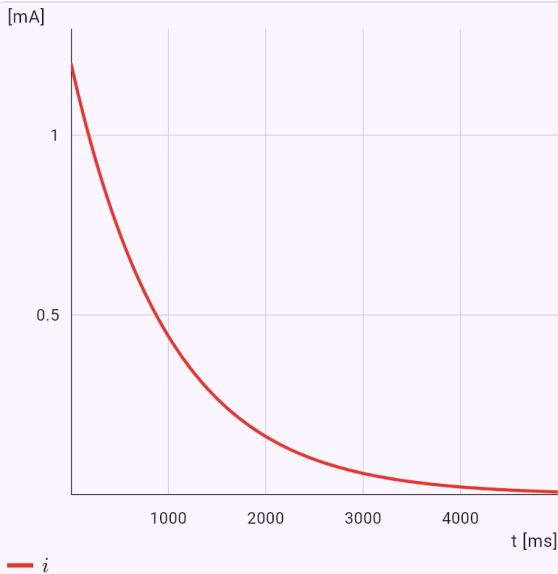
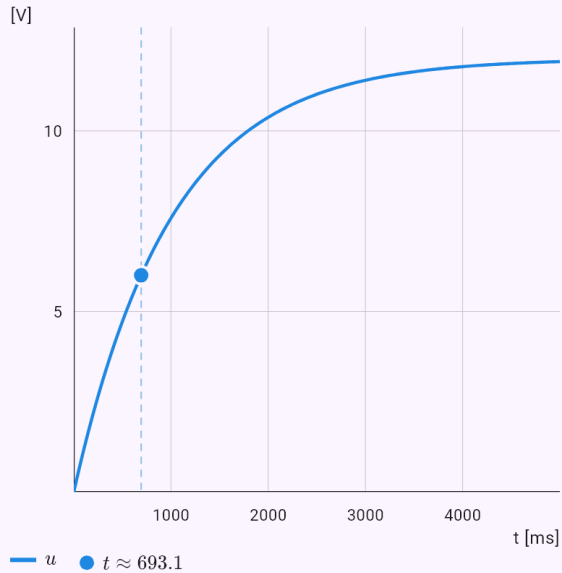
$$t \approx 693 \text{ ms}$$

Steps

- Find t ; given
 $R = 10 \text{ k}\Omega$, $C = 100 \mu\text{F}$, $U = 12 \text{ V}$, $u = 6 \text{ V}$
- Charging a capacitor through a resistor, Kirchhoff's law gives $U = Ri + u$, and the current charges the capacitor: $i = C \cdot du/dt$. Together $RC \cdot du/dt + u = U$. The voltage approaches the source's: $u = U(1 - e^{-(t/\tau)})$, $\tau = RC$ being the time constant; the current $i = (U/R) \cdot e^{-(t/\tau)}$ decreases. When discharging, $u = U_0 e^{-(t/\tau)}$. The model's equation: $RC \frac{du}{dt} + u = U$
- Time constant: $\tau = RC = 10 \cdot 100 \approx 1000 \text{ ms}$
- Separating the variables: $\frac{du}{u - U} = -\frac{dt}{\tau}$
- Integrating both sides: $\ln |u - U| = -\frac{t}{\tau} + C$
- Exponentiating – the constant $\pm e^C$ is called C again:
 $u = U + C e^{-\frac{t}{\tau}}$
- Substituting the initial condition $u(0) = 0$:
 $0 = U + C \Rightarrow C = -U$
- The solution of the initial condition (the particular solution): $u = U(1 - e^{-\frac{t}{\tau}})$

- Expressing the sought quantity: $t = -\tau \ln \left(1 - \frac{u}{U}\right)$
- Substituting: $t = -1000 \ln \left(1 - \frac{6}{12}\right) \approx 693.1 \text{ ms}$

Graph



The model's quantities in time, computed from the solution; the answer is marked.

A damped spring – the displacement at 1 s.

Input: `model(spring; find=x[cm]; m=0.5[kg]; k=20[N_m]; b=1[kg_s]; x0=5[cm]; v0=0[m_s];`

$t=1[s])$

Result

$$x \approx 1.827 \text{ cm}$$

Steps

- Find x ; given

$$m = 0.5 \text{ kg}, \quad k = 20 \text{ N/m}, \quad b = 1 \text{ kg/s}, \quad x_0 = 5 \text{ cm},$$

- Converting the units: $x_0 = 5 \text{ cm} = 0.05 \text{ m}$

- A body on a spring feels the spring's force $-kx$ (Hooke's law), damping $-b \cdot v$ (a resistance proportional to the speed) and possibly a driving force $F_0 \cdot \cos \omega t$. By Newton's 2nd law $m \cdot x'' = -kx - b \cdot x' + F_0 \cdot \cos \omega t$, that is $m \cdot x'' + b \cdot x' + kx = F_0 \cdot \cos \omega t$ – a linear equation of the 2nd order with constant coefficients. The roots of the characteristic equation tell whether the body oscillates (underdamping) or just returns to the equilibrium. Driving near the natural frequency $\omega_0 = \sqrt{k/m}$ causes resonance. The model's equation:

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

- Natural angular frequency (without damping):

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{20}{0.5}} \approx 6.325 \text{ s}^{-1}$$

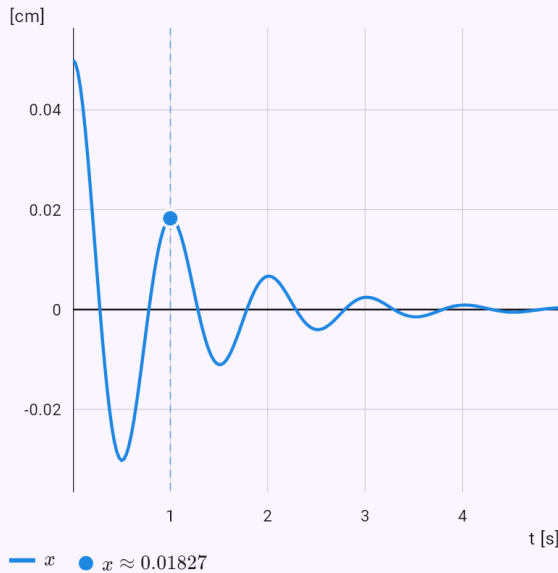
- Damping rate: $\delta = \frac{b}{2m} = \frac{1}{2 \cdot 0.5} \approx 1 \text{ s}^{-1}$

- We look for solutions $x = e^{\lambda t}$; substituting and dividing by $e^{\lambda t}$ gives the characteristic equation:

$$0.5\lambda^2 + \lambda + 20 = 0$$

- The roots of the characteristic equation:
 $\lambda_{1,2} = -1 \pm 6.245i$
- Underdamping ($\delta < \omega_0$; $\delta = 1$, $\omega_0 = 6.325$): complex roots – damped oscillations.
- The homogeneous equation's solution:
 $x_h = e^{-t}(C_1 \cos 6.245t + C_2 \sin 6.245t)$
- We substitute the conditions into the general solution:
 $x(0) = 0.05$, $x'(0) = 0$, hence
 $C_1 \approx 0.05$, $C_2 \approx 0.008006$
- The solution of the initial condition (the particular solution):
 $x = e^{-t}(0.05 \cos 6.245t + 0.008006 \sin 6.245t)$
- Substituting: $x(1) \approx 0.01827 \text{ m}$
- Converting the units: $x \approx 0.01827 \text{ m} = 1.827 \text{ cm}$

Graph



The model's quantities in time, computed from the solution; the answer is marked.

An epidemic with $R_0 = 2.5$ – the peak and the final number of infected.

Input: `model(sir; find=overview[d]; N=10000[pcs]; I0=10[pcs]; R0=2.5; D=7[d]; rates=r0)`

Result

$$I_{max} \approx 2340 \text{ ind.}$$

$$t_{max} \approx 34.1 \text{ d}$$

$$R_{\infty} \approx 8930 \text{ ind.}$$

Steps

- Find t_{max} ; given
 $N = 10\,000 \text{ ind.}, \quad I_0 = 10 \text{ ind.}, \quad R_0 = 2.5, \quad D = 7 \text{ d}$

- The population N is split into the susceptible S , the infected I and the recovered (immune) R . Infections increase in proportion to the encounters of the susceptible with the infected: $\beta \cdot S \cdot I / N$; the infected recover at the rate $\gamma = 1/D$, D being the illness' duration. The system $S' = -\beta SI/N$, $I' = \beta SI/N - \gamma I$, $R' = \gamma I$ has no solution in elementary functions – it is solved numerically. Exactly, though, we derive the basic reproduction number $R_0 = \beta/\gamma$, the epidemic's peak, the final number of infected and the herd immunity threshold $1 - 1/R_0$. The model's equation

$$S' = -\frac{\beta SI}{N}$$

$$I' = \frac{\beta SI}{N} - \gamma I$$

$$R' = \gamma I$$

- Recovery rate: $\gamma = \frac{1}{D} = \frac{1}{7} \approx 0.1429 \text{ d}^{-1}$

- Transmission rate: $\beta = R_0 \gamma \approx 0.3571 \text{ d}^{-1}$

- The sum $S + I + R = N$ is constant (the equations add up to zero).

- Herd immunity threshold (the immune share at which the illness stops spreading): $1 - \frac{1}{R_0} \approx 60\%$

- The epidemic takes off, since initially

$$I' > 0 \Leftrightarrow S > \frac{N}{R_0} \approx 4000$$

- The epidemic peaks at $S = N/R_0$; from the equation for dI/dS

$$\frac{dI}{dS} = -1 + \frac{N}{R_0 S} \Rightarrow I + S - \frac{N}{R_0} \ln S = C \Rightarrow I_{max} = I$$

- Solving the equation numerically (bisection)

$$\ln \frac{S_{\infty}}{S_0} = \frac{R_0(S_{\infty} - N)}{N} \Rightarrow S_{\infty} \approx 1072 \text{ ind.}$$

- In total the illness affects:

$$R_{\infty} = N - S_{\infty} \approx 8928 \text{ ind.} \quad (89.3\%)$$

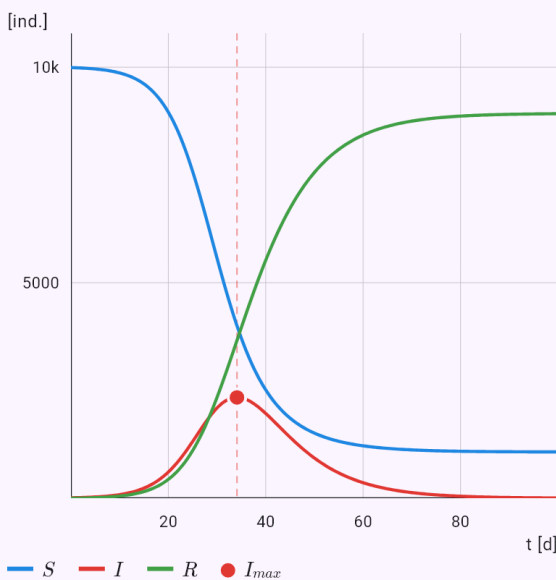
- The system can't be solved in elementary functions – it's solved numerically by the 4th-order Runge–Kutta method with the step 0.05 on $[0, 100]$

- From the numerical solution: $t_{max} \approx 34.1 \text{ d}$

Graph

In time

Phase plane



The model's quantities in time, computed from the solution; the answer is marked.

Predator and prey – the equilibrium, its stability and the period.

Input: `model(predator; find=overview[mo]; a=0.5[per_mo]; b=0.01[per_pcs_mo]; c=0.4[per_mo]; d=0.005[per_pcs_mo]; x0=100[pcs]; y0=20[pcs])`

Result

$$[x^*, y^*] \approx [80, 50]$$

$$T \approx 14.9 \text{ mo}$$

Steps

- Find T ; given

$$a = 0.5 \text{ mo}^{-1}, \quad b = 0.01 (\text{ind.} \cdot \text{mo})^{-1}, \quad c = 0.4 \text{ mo}^{-1}$$

- Prey x multiplies without predators (ax), predators y die out without prey ($-cy$). Encounters, proportional to the product xy , take prey away ($-bxy$) and give predators food ($+dxy$). The system $x' = ax - bxy$, $y' = -cy + dxy$ (Lotka, Volterra) has the equilibrium $[c/d; a/b]$. Linearized there it has purely imaginary eigenvalues – the populations oscillate around the equilibrium, the predators lagging behind the prey. It's solved numerically; the equilibria and the first integral are derived exactly. The model's equation

$$x' = ax - bxy$$

$$y' = -cy + dxy$$

- Equilibria – all the derivatives are zero

$$x(a - by) = 0, \quad y(dx - c) = 0 \Rightarrow [0, 0], \quad \left[\frac{c}{d}, \frac{a}{b}\right] \approx [80, 50]$$

- The stability by linearization – the system's Jacobian matrix

$$J = \begin{pmatrix} a - by & -bx \\ dy & dx - c \end{pmatrix}$$

- At $[0, 0]$ eigenvalues $\lambda_1 = -0.4$, $\lambda_2 = 0.5$ – a saddle (unstable).

$$\begin{pmatrix} 0.5 & 0 \\ 0 & -0.4 \end{pmatrix}$$

- At $[80, 50]$ eigenvalues $\lambda_{1,2} = \pm 0.4472i$ – a centre (oscillations around the equilibrium).

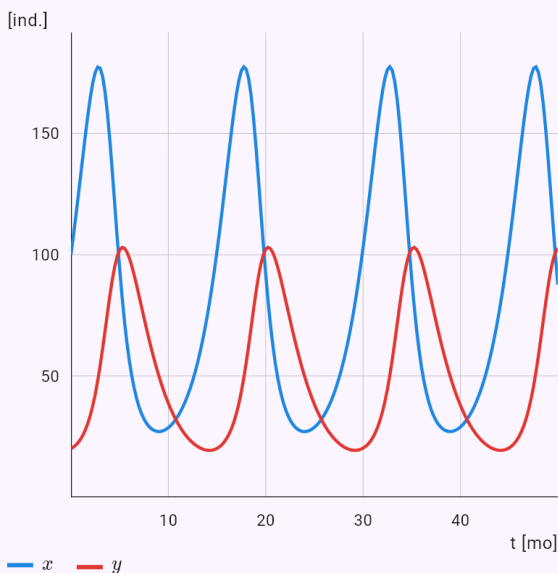
$$\begin{pmatrix} 0 & -0.8 \\ 0.25 & 0 \end{pmatrix}$$

- Dividing the equations and separating gives a first integral – a quantity constant along a trajectory:
 $V = dx - c \ln x + by - a \ln y = C$
- Since V doesn't change along a trajectory, the trajectories are closed curves – the populations oscillate periodically around the equilibrium.
- Period of small oscillations around the equilibrium (from the eigenvalues): $T \approx \frac{2\pi}{\sqrt{ac}} \approx 14.05 \text{ mo}$
- The system can't be solved in elementary functions – it's solved numerically by the 4th-order Runge-Kutta method with the step 0.0125 on $[0, 50]$
- From the numerical solution: $T \approx 14.95 \text{ mo}$

Graph

In time

Phase plane



The model's quantities in time, computed from the solution; the answer is marked.

Coming soon

We are working on these types of problems or have them planned. The order may change depending on what you need most.

Equations and inequalities

- Incomplete quadratic equations by factoring or taking roots ($x^2 = 9$) instead of the discriminant.

Functions, derivatives and integrals

- Integral applications with parameters – general formulas (a sphere of radius r , an ellipse, a cone).
- The shaded area and the 3D solid for parametric and polar curves.
- Physical applications of derivatives and integrals (velocity, acceleration, work, centre of mass).
- The complex form of the Fourier series and the spectrum.
- The Riemann and Lebesgue integrals of special functions (the Dirichlet and Riemann functions).

Differential equations

- Systems of differential equations given as equations (the applications have them already) and their phase portrait.
- The Laplace transform with the Heaviside function, shifts and periodic right sides.

Other areas

- Vectors – magnitude, dot, cross and triple products, angles, areas and volumes.
- Matrices – operations, determinant, rank, inverse matrix, eigenvalues and eigenvectors.
- Construction problems – triangles, quadrilaterals and circles from given elements, the construction steps and the number of solutions.
- Complex numbers in the polar and exponential form, de Moivre's theorem, n -th roots.
- Cramer's rule for systems of linear equations.
- Functions of several variables, multiple, line and surface integrals.
- Exact and numerical result modes (decimals instead of fractions and roots).
- Word problems – setting up the equations from the text and solving them.